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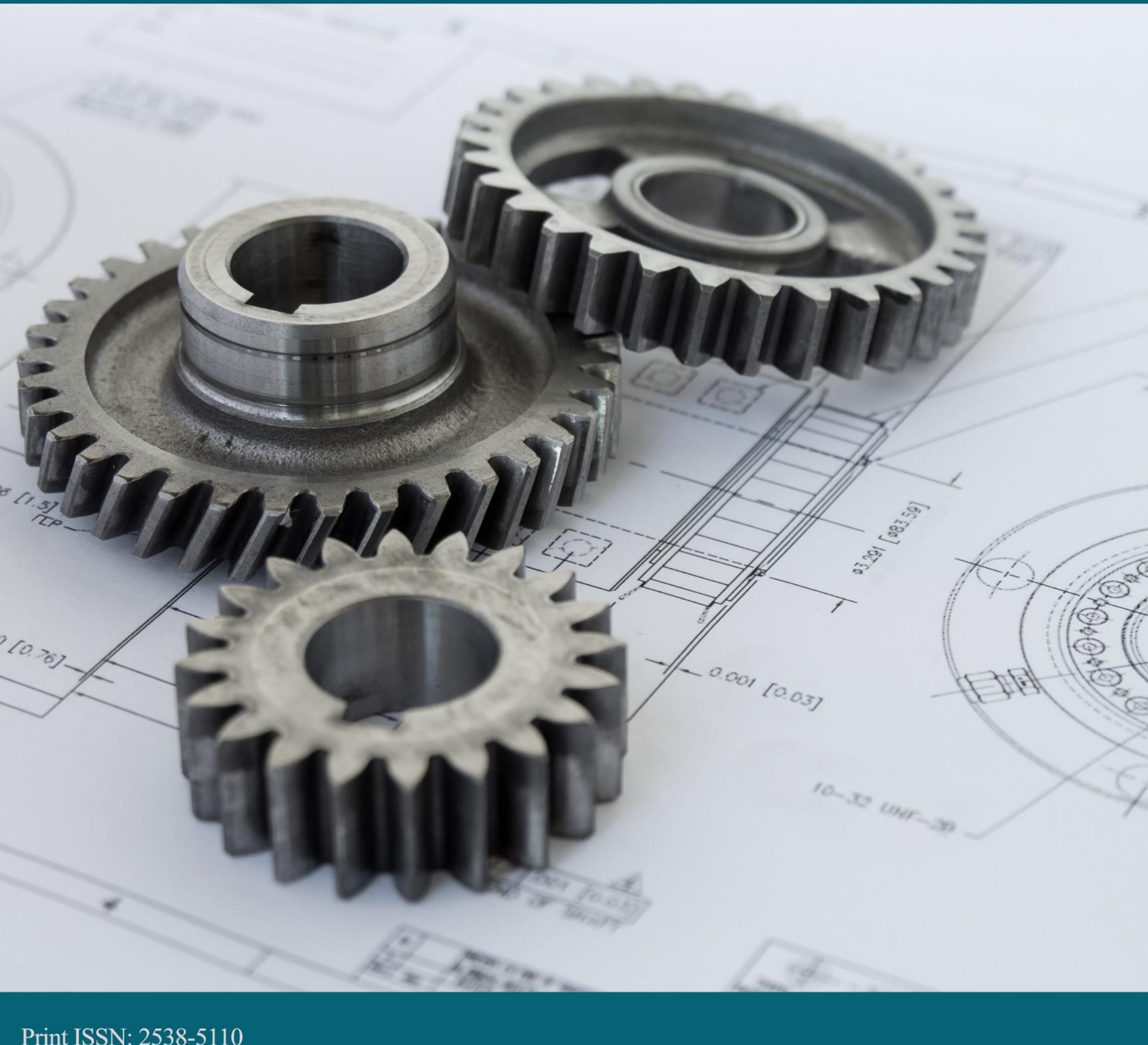

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Multi-Depot Electric Vehicle Routing Problem with Fuzzy Time Windows and Pickup/Delivery Constraints

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A B S T R A C T

The use of an Electric Vehicle (EV), particularly in different operations of goods distribution is a solution for salvaging the crowded cities of the world from air and noise pollutions as well as Green House Gas (GHG) emission. This paper presents a Multi-Depot Electric Vehicle Routing Problem (MD-EVRP) with recharging stations by considering the expected penalty of fuzzy time windows in pickup/delivery. Since the MD-EVRP with Fuzzy Time Windows and Pickup/Delivery (MD-EVRP-FTW-PD) constraints is an NP-hard problem, three meta-heuristics (i.e., Simulated Annealing (SA), Variable Neighborhood Search (VNS) and a hybrid of SA and VNS (VNS-SA)) are used to solve such a hard problem. The parameters of these algorithms are measured by the Taguchi experimental design method. The proposed hybrid VNS-SA algorithm is more efficient in comparison with other algorithms.

1. Introduction

Air pollution is the main problem of general health in developing countries. The causes of the many such pollutions are rooted in the energy section. It is the fourth and the greatest threat to human health after high blood pressure, unhealthy date, and smoking [1]. Almost 6.5 million people's annual death are assigned to the low quality of the air.

Earth warming caused by the emission of a lot of greenhouse gas is a global concern [2]. The second participation in Green House Gas (GHG) with 23% of CO_2 emission in 2014 is a transportation section [3]. This section depends severely on the combustion of oil so improving and changing the energy consumption in this industry is one of the national aims of many countries. Countries as India, England, and Germany declared that they will not sell and use the vehicle with inner combustion to 2030 [4].

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European Union obligates to reduce CO_2 in 2050 in comparison with 1990 between 80 and 90%. To reduce GHG emission and keep the global warming increase under 2 centigrade [5].

An electric vehicle is a helpful approach to solve the climate problem [6]. Among vehicles, electric trucks with a medium-size in comparison with a diesel truck produce less CO_2, 300 tons on average [7]. Although, most of the scientific and technological achievements related to the private and general electric vehicles, in recent years, a large virtue of research is concerning by using electric vehicles in the distribution of goods. In the same portion, the share of the electric vehicle market is increasing. For example, in China, the sale of these vehicles has reached 331000 per year to more than 50% between 2015 and 2017, which is 777000 set [8]. Also, in the world, 1.26 milliard electric vehicles in 2015 are used that are 100% more than 2010 [9]. This increase is due to features with an environment as a lack of GHG emission, less noise pollution, and high random energy that could provide logistic companies with a green picture for increasing social consciousness and environmental awareness [8]. However, this share of the market for vehicles is not satisfied. Then with financial incurring politics to finance the cost of buying a car, road toll exemption, and priorities of reaching the city center that some of the countries use it, can help to use these vehicles widely [10-13].

An electric vehicle challenges a limitation of movement ranges, and longings of complete charge time. Movement ranges limitation of these vehicles, that is, reaching to customers and returning to depot needs to recharge the battery, then the place of recharge station is important. Because of the long duration of charging time, it uses a changing battery technique instead of charging the battery, to spend less time. In the way vehicles to complete a route, faced trouble because of lack of suitable and enough under structures for changing the battery and lack of similar standards. For this reason, logistics companies prefer to provide these facilities by themselves. Thus, the place of the recharging stations plays an important role in optimization.

A fuzzy optimization method is widely used in Vehicle Routing Problems (VRPs); however, a very limited uncertain problem is proposed for electric vehicles. To deal with unknown parameters, some researchers have used an incidental optimization method. However, in practical applications, it is difficult to describe these parameters as incidental variations for the lack of enough historical data for analyzing. In return, it can be used fuzzy variations to deal with these unknown parameters [14].

The model proposed in this paper for the Multi-Depot Electric Vehicle Routing Problem with Fuzzy Time Windows and Pickup/Delivery (MD-EVRP-FTW-PD) is time windows of fuzzy service time. Then, the model is first solved by GAMS software for small-sized problems and by three meta-heuristic algorithms (i.e., Simulated Annealing (SA), Variable Neighborhood Search (VNS) and a hybrid of SA and VNS (VNS-SA)) coded in MATLAB software for large-sized problems. Finally, the efficiency of the algorithms and solution time is reported and compared. Experimental results show that the proposed algorithm is effective for solving the MD-EVRP-FTW-PD.

This paper is organized as follows. Section 2 considers the related work concerns with this investigation. Section 3 presents the MD-EVRP-FTW-PD model. Section 4 proposes meta-heuristic algorithms for solving this model. Section 5 reports the findings and result analyses. Finally, Section 6 provides the conclusion and some future studies.

2. Literature Review

The wishes of the countries to use electric vehicles not only in cities with high crowded population and pollution but also in logistic distribution networks are increasing. Although in an electric vehicle section, there are many mathematical models, in commercial electric vehicles and trucks, there are many fields to investigate. First, a brief literature review of VRPs with Pickup and Delivery (VRP-PD) and fuzzy theory in VRPs is presented. Then, the papers related to EVRPs are reviewed to find the research gap.

Çatay [15] studied on the VRP-PD. Fan [16] presented the VRP-PD and time windows to increase customer satisfaction. Wang and Qiu [17] studied the VRP-PD and probable demand and solved their model by a meta-heuristic algorithm. Setak et al. [18] presented a multi-depot capacitated Location-Routing Problem (LRP) with simultaneous pickup and delivery and split loads. They used heterogeneous vehicles in their model and solved the problem with a generic algorithm. Wang and Chen [19] studied the VRP-PD and time windows and solved their problem by a generic algorithm. Several papers in the VRP-PD and time windows or multi-depot can be found in [20-24]. The fuzzy theory has also been used widely in VRPs, such as LRP [25], multi-depot VRP for hazardous material [26], fuzzy multi-depot VRP-PD [27]. Other related VRPs can be found in [28-30].

Although, the purpose of the VRP and EVRP is to find an optimal routing to cover all of the customers' demands. As mentioned before, in contrast to classical vehicles, the domain of driving electric vehicles is short and the limits will be more due to battery capacities.

Artmeier et al. [31] used a graph theory concept and suggested a routing optimization problem with alternative fuel vehicles. They presented the shortest path algorithm by considering vehicle limitations. Other studies can be mentioned in the literature, such as a study on the battery capacity by Lin et al. [32], concentration on variable policies, such as type of charges by Mart´ınez-Lao et al. [33], partial recharge station electric vehicle with time windows by Keskin and Çatay [34] and the possibility of battery swapping by Yang and Sun [35].

The other studies concerning an EV location and routing problem are also presented. Schiffer and Walther [36] and Li et al. [37] studied the charge station LRP. Hof et al. [38] presented swapping battery and charge station LRP. Paz et al. [39] considered multi-depot LRP in three models. The first model has only a partial charge station in the route. The second model has only a battery-swapping station and the third one has partial charge station and battery swapping in the route. The results of their model show that partial charge and battery swapping are more efficient. Keskin and Çatay [40] represented the EVRP with time windows and fast chargers and proposed the ALNS algorithm to solve the model. For further studies in this field, the reader can refer to [41].

The point is this that the above-mentioned studies have been done in a certain environment and the unknown factors affecting the effects have been ignored suggested to these unknown factors. In fact, in addition to unknown factors relating to vehicles, many of the city and intercity logistic applied programs are the cases out of the atmosphere, traffic condition, road conditions, the existence of recharging stations, or battery changing, and unknown demands of customers. Although Zhang et al. [9] regarded three parameters of time service, energy consumption, and travel time as fuzzy, their model was one depot.

The model presented in this paper has a multi-depot, a set of customers with certain demand, the possibility of pickup/delivery, charge stations, and homogenous fleet with a fixed capacity of the EV. In this model, time windows of fuzzy service time for customer satisfaction. This model is very suitable for the decision-maker considering a multi-depot distribution and determining the route with the minimum cost under uncertainty.

The contributions of this paper are to develop the EVRP-TW-PD model to a multi-depot model and use the expected penalty of time windows with a fuzzy approach for salsify customers. The model is solved for small and large sizes with GAMS software and also three algorithms of SA, VNS, and combination of SA and VNS algorithm.

3. MD-EVRP-FTW-PD Model

In this section, the important assumptions for formulating the model is first presented, and then, it concerns with notations and a description of the model.

3.1. Assumptions Model

The main assumptions about the proposed model are as follows:

- The numbers of depots and vehicles in each inventory are specified.
- Electric vehicles have the same capacity loading and battery as well as move with constant speed a limitation for driving.
- The electrical energy consumption multiple is constant and is adjusted to the distance.
- Recharging time in the station is fixed.
- EVs move from distribution depots or recharging stations with a complete charge and do not consume energy in a customer service process.
- Each customer has only one specific demand (delivery or pickup), and services to each customer are provident only one time.
- EVs exit each disturbing depot, then provide service to customers continuously and finally return to the same depot from where it had started to move.
- To satisfy customers, the time windows of a fuzzy number are considered to pick up or delivery.

3.2. Notations

The following notations are used in the proposed MD-EVRP-FTW-PD shown in (e.g., *Table 1*):

Table 1. *Definitions of sets, parameters, and variables.*

Notations	Definition
Sets and Indices	
O	*Set of all depots; denoted by index o.*
NC	*Set of customers.*
NE	*Set of recharging stations.*
NT	*Set of all nodes,* $O \bigcup NC \bigcup NE$*, i, j= {1,2, 3, …, n}.*
NK	*Set of electric vehicles; denoted by index k.*

Notations	Definition
Parameters	
Dis_{ij}	*Distance between nodes i and j.*
Dem_i	*Demand of node i.*
Cap_k	*Capacity of electric vehicle k.*
$[LT_i, UT_i]$	*Fuzzy interval: lower and upper bounds of acceptable time windows of node i.*
$[LE_i, UE_i]$	*Fuzzy interval lower and upper bounds of expected time windows of node i.*
ST_i	*Service time in node i.*
TT_{ij}	*Travel time between nodes i and j.*
BC	*Battery capacity of electric vehicle k of node i.*
EC	*Electricity consumption coefficient.*
CU	*Cost of unit transportation.*
CW	*Cost of unit time for waiting.*
CL	*Cost of unit time cost for late.*
K_o	*Number of electric vehicle k of depot o.*
LN	*An arbitrarily large number.*
Variables	
X_{ijk}	*1 if the route between nodes i and j is traveled by electric vehicle k; 0, otherwise.*
Y_{ik}	*1 if node i is served by electric vehicle k; 0, otherwise.*
W_{ok}	*1 if electric vehicle k travels from depot o; 0, otherwise.*
A_{ijk}	*Amount of pickup or delivery electric vehicle k on board on nodes i and j.*
TA_{ik}	*Time of vehicle k arrives at node i.*
TL_{ik}	*Time of vehicle k leaves node i.*
TS_{ik}	*Service start time of vehicle k at node i.*
RCA_{ik}	*Remaining charge when vehicle k arrives at node i.*
RCL_{ik}	*Remaining charge when vehicle k leaves node i.*
Sub_{ik}	*Variable used for elimination of sub-tours.*

3.3. Determination of the Objective Function and Constraints

In this paper, the minimization objective function is divided into two sections: Transporting cost and penalty cost of time windows. These sections are the total cost of transportation and the penalty cost of violating of soft and hard time windows, respectively. Since it is very important to rank the fuzzy subsets [42] in the transportation problem, the service time windows are assumed to be fuzzy. Different methods for ranking of fuzzy subsets are proposed in the literature. In this paper, the developed method of Liou and Wang [43] is used. Based on this method, the total integral value is a convex combination of the right and left integral values through an index of optimism, $\beta \in [0, 1]$. The left and right integrals show the optimistic and pessimistic points of view of the customers, respectively. The total integral value is reached by a convex combination of the right and left integral values. This value is used to rank fuzzy numbers.

In this model, uncertain parameters, the pattern sets of the fuzzy number $F= (a, m, b)$, where a, m and b are estimated as the pessimistic value and most likely, and optimistic values, respectively *Eq. (1)*. If F is a triangular fuzzy number, the membership function S_F will be defined as follows [42, 44-46].

$$S_F(x)=\begin{cases} f_n(x)=\frac{x-a}{m-a} & \text{if } (a\le x\le m) \\ 1 & \text{if } (x=m) \\ g_n(x)=\frac{b-x}{b-m} & \text{if } (m\le x\le b) \\ 0 & \text{if } (x\le a, x\ge b) \end{cases} \quad (1)$$

Then, the Expected Interval (EI) and the Expected Value (EV) of *F* are calculable by:

$$EI(F)=[E_1^{\ n},E_2^{\ n}]=[\int_0^1 f_n^{-1}(x)dx, \int_0^1 g_n^{-1}(x)dx]=[\frac{1}{2}(a+m),\frac{1}{2}(m+b)]. \quad (2)$$

$$EV(F)=\frac{E_1^{\ n}+E_2^{\ n}}{2}=\frac{a+2m+b}{4}. \quad (3)$$

Therefore, for each of the parameters *LTi*, *UTi*, *LEi*, and *UEi*, three *Pessimistic (A)*, *probabilistic (m)*, and *Optimistic (b)* times are assumed. For example, *LTi*: *LT (a)i*, *LT (m)i*, *LT (b)i*. Then, we calculate the penalty cost of the mixed time windows if the vehicle arrives earlier or later than these intervals. The other limitation of this model is the limitation of a vehicle if the vehicle is not charged to reach the customer, it should be returned to the recharge station if not, it should be returned to the depot from which the vehicle had started to move.

3.4. Mathematical Model

In this subsection, the presented model is as follows:

$$\text{Minimize } Z=CU(\sum_{i\in NT}\sum_{\substack{i\in NT\\ j\neq i}}\sum_{k\in NK} Dis_{ij}X_{ijk})+\sum_{i\in NC}\sum_{k\in NK} Y_{ik}(CW.LL_{ik}+CL.UL_{ik}) \quad (4)$$

$$\text{subject to } \sum_{i\in NT, i\neq j}\sum_{k\in NK} X_{ijk}=1, \qquad \forall j\in NC, \quad (5)$$

$$\sum_{i\in NT, i\neq j}\sum_{k\in NK} X_{jik}=1, \qquad \forall j\in NC, \quad (6)$$

$$\sum_{j\in NC\cup NE} X_{ojk}=\sum_{j\in NC\cup NE} X_{jok}\le W_{ok}, \qquad \forall k\in NK, o\in O, \quad (7)$$

$$\sum_{i\in NT, i\neq j} X_{ijk}=\sum_{i\in NT, i\neq j} X_{jik}=Y_{jk}, \qquad \forall j\in NC\cup NE, k\in NK, \quad (8)$$

$$X_{ojk}=0 \qquad \forall j\in O\cup NE, k\in NK, o\in O, \quad (9)$$

$$0\le A_{ijk}\le Cap_k.X_{ijk} \qquad \forall i,j\in NT, j\neq i, k\in NK, \quad (10)$$

$$\sum_{i\in NT, i\neq j}\sum_{k\in NK} A_{ijk}-Dem_j=\sum_{i\in NT, i\neq j}\sum_{k\in NK} A_{jik} \qquad \forall j\in NC, \quad (11)$$

$$TL_{ok}=0 \qquad \forall k\in NK, o\in O, \quad (12)$$

$$TA_{jk}=\sum_{i\in NT, i\neq j} X_{ijk}.(TL_{ik}+TT_{ij}) \qquad \forall j\in NC\cup NE, k\in NK, \quad (13)$$

$$TL_{ik}=Y_{ik}.(SST_{ik}+ST_i) \qquad \forall j\in NC\cup NE, k\in NK, \quad (14)$$

$$SST_{ik}=Y_{ik}.(TA_{ik}+LL_{ik}) \qquad \forall j\in NC\cup NE, k\in NK, \quad (15)$$

$$RCA_{ik} \geq 0 \qquad \forall j \in NT, k \in NK, \tag{16}$$

$$RCA_{jk} = \sum_{i \in NT, i \neq j} X_{ijk}.(RCL_{ik} - EC.Dis_{ij}) \qquad \forall j \in NT, k \in NK, \tag{17}$$

$$RCL_{ik} = BC \qquad \forall i \in O \bigcup NE, k \in NK, \tag{18}$$

$$RCL_{ik} = RCA_{ik} \qquad \forall j \in NC, k \in NK, \tag{19}$$

$$Sub_{ik} - Sub_{jk} + LN.X_{ijk} \leq LN - 1 \qquad \forall j \in NT, j \in NC, j \neq i, k \in NK, \tag{20}$$

$$\sum_{j \in NC \bigcup NE} \sum_{k \in NK} X_{ijk} \leq K_o \qquad \forall i \in O, \tag{21}$$

$$TA_{ik} \geq LT_i \qquad \forall i \in NC \bigcup NE, k \in NK, \tag{22}$$

$$TA_{ik} \leq UT_i \qquad \forall i \in NC \bigcup NE, k \in NK, \tag{23}$$

$$X_{ijk} \in \{0,1\} \qquad \forall i, j \in NT, i \neq j, k \in NK, \tag{24}$$

$$Y_{ik} \in \{0,1\} \qquad \forall j \in NT, k \in NK, \tag{25}$$

$$W_{ok} \in \{0,1\} \qquad \forall o \in O, k \in NK, \tag{26}$$

$$Sub_{ik} \geq 0, LN \geq 0 \qquad \forall i \in NT, k \in NK. \tag{27}$$

Constraint (4) illustrates the objective function representing the total cost, including transportation costs and the costs for violating time windows. *Constraints (5)* & *(6)* ensure that every customer is served exactly once. *Constraint (7)* guarantees that each vehicle moved from each depot should be returned to the same depot. *Constraint (8)* represents the vehicle flow-conservation equation, that is, the number of times vehicle k enters into a point *i* is equal to the number of times it leaves point *i*. *Constraint (9)* guarantees the vehicle do not go to recharge stations from depots directly. *Constraint (10)* imposes the maximum vehicle capacity constraint. *Constraint (11)* ensures the vehicle load variation on a route. *Constraint (12)* imposes the departure time of a vehicle from each depot. *Constraints (13)* - *(15)* are to calculate the time that vehicle *k* arrives at and departs from the point *i* and the time starting service at point *i*. *Constraint (16)* ensures that the vehicles are fully charged after departing from the distribution center or the recharging stations. *Constraints (17) - (19)* represent the electricity constraint. *Constraint (20)* guarantees the elimination of the sub-tour. *Constraint (21)* represents the number of vehicles that cannot exceed m_o. *Constraints (22)* & *(23)* represent time windows (according to what is said, fuzzy numbers). *Constraints (24)* & *(26)* are binary variables. *Constraint (27)* guarantees Sub_{ik}, where *LN* is a non-negative auxiliary variable.

3.5. Model Linearization

Considering the nature of the problem, the model is formulated in nonlinear form. By considering that nonlinear models are time-consuming in comparison with their solving linear models and there is not available an algorithm for ensuring a comprehensive optimal answer. The presented mathematical model is a Mixed-Integer Nonlinear Programming (MINLP) model due to some nonlinear equations. The multiple variables in *Eqs. (4), (13)-(15)* & *(17)*. Linearization methods are now applied to obtain an equivalent linear mathematical model. Linear Programming (LP) as a very multi-functional technique is used to making the design and solving a variety of problems [47]. This method was proposed by McCormick [48].

In the second term of the objective function, two variables change *Eqs. (28)* and *(29)* are occurring. Some equations are non-linear that will be changed in linear equations. Then, the model is solved in an MIP form.

$$LL_{ik} = \max\left[\left(LE_i - TA_{ik}\right), 0\right] \quad \forall i \in NT, k \in NK. \quad (28)$$

$$UL_{ik} = \max\left[\left(TA_{ik} - UE_i\right), 0\right] \quad \forall i \in NT, k \in NK. \quad (29)$$

4. Proposed Method

A reason for using the meta-heuristic algorithm is the history and complexity of the MD-EVRP-FTW-PD model for its optimization. This section represents a solution representation method, neighborhood structure, solution initialization, operators to deal with the infeasible solutions, and adjusted method of parameters and parameters of each algorithm. Then, three meta-heuristic algorithms (i.e., SA, VNS, and a hybrid of SA and VNS) are proposed to solve the presented model.

4.1. Solution Representation Method

To write the computer programs for the proposed algorithms, we first need the representation structure method of solutions. The structure must be in a manner that can be observed and studied performing a solution. The structure design of the representation method has an important role in the fitness function. The solution representation is a solution string that includes several cell customers, depots, and charge stations. Vehicle order of visit between nodes is determined in this representation. The (e.g., *Fig. 1*) shows a small problem with four customers and two depots and one recharge station are shown. In this figure, the places of 1 and 2 are depots, and 3, 4, 5, and 6 are nodes of customers, and 7 is the charge station showing an electric vehicle movement tour. To assign customers to electric vehicles, all the cells from the beginning of the string will be assigned to the first vehicle until we reach a zero value.

1	3	4	5	6	1	0	7	2

Fig. 1. *Example of a solution representation*.

4.2. Neighborhood Structure

The main purpose of the neighborhood structure in the algorithms is to construct a neighborhood solution out of an available solution by changing them [49]. The SA algorithm uses temperature changes to make neighborhood structures. The VNS algorithm uses three moving actors to make a new neighborhood for current solutions. The hybrid VNS-SA algorithm uses the combination of these structures presented separating in describing each algorithm.

4.3. Solution Initialization Algorithm

To begin the algorithm, it needs to initialize the suitable and quality solution. After recalling the main parameters out of initializing the solutions based on the demand limitation (for delivery and pickup), the maximum limit of the loading capacity (the algebraic sum of delivery and pickup should not exceed the vehicle capacity) and the charge limitation of EVs should have enough charge. Breach of limitations may result in an infeasible solution thus the strategy of the punishment extent suitable with the extent of a limitation breach in the algorithm should be added to the fitness function. These parameters (i.e., Sol.demand, Sol.charge, and Sol.cap) are equal to the extent of punishment for the demand limitation breach of per unit, charge, and the extent of the loading capacity. This results in a feasible solution with the β coefficient.

4.4. Simulated Annealing Algorithm

SA is a simple and effective meta-heuristic algorithm for solving optimization problems proposed by Metropolis in 1953. It was first used in optimization by Kirkpatrick et al. (1983). SA has been used successfully in various types of VRPs and shown good performance [50].

SA is an algorithm based on slowly annealing technique, which is based on local searching in solution space and accepts probabilistic and non-standard solutions to escape from local optimization to reach for better answers. By decreasing the annealing condition for each given temperature, the level of energy is calculated according to the Boltzmann distribution. Algorithm parameters are included in initial temperature, final temperature, the number of iterations in the fixed temperature, and the number of iterations in the case of lack of any solution improvement. The algorithm by establishing one of two conditions: reaching to final temperature or several important solution iterations is concluded. To escape of local optimization, SA uses a function for accepting the worse solution.

In this probable function, the current situational situation, a situation in the neighborhood, Boltzmann consonant, and current temperature are determined. In *Eq. (30)*, Δ means the deviation of energy (i.e., objective) function values.

$$R = \exp\ [-\Delta/(K * T)]. \quad (30)$$

It is necessary to mention that the initial solution of the SA algorithm is presented by using the method of producing the initial solution in the next section.

In SA, one of neighborhood search methods is used in each repetition, until the best solution results; however, there is not any specific procedure for using the neighborhood search method. This neighborhood search process will be continued until the number of repetition reach to determined quantity, then the system temperature will be reduced and this process will be continued to reach the stopping criterion.

4.5. Variable Neighborhood Search Algorithm

VNS is a meta-heuristic algorithm used to solve hybrid and global optimization problems. In 1997, Mladenović and Hansen [51] first developed a VNS algorithm. The main idea of this algorithm is the systematic neighboring change by using a local searching method. The general structure of this method is as follows. First, several neighboring structures and a primary answer are determined, then the algorithm using the primary neighboring structure, based on two first improvement selection approaches or best improvement selection starts to search. In the case of observing any improvement in solution, the better solution is replaced by the current solution, and the neighboring structure returns to the first start. After several iterations and in a case of not observing any improvement in answer, it enters the next neighboring structure. The stopping condition of this algorithm is included in the case as a limitation in the number of iterations, the computational time, the number of iterations between two consequent improvements in answering as well finding a locally optimal solution in the whole neighboring structure. The VNS algorithm has a more meta-heuristic algorithm structure and its advantage is fewer parameters, consequently, it has high speed.

The suggested VNS algorithm to the given problem has four main phases, namely the generate solution initialization, shaking, local search phase, and altimetry stop.

- Generating the initial solutions: The solution initialization is made with generating a line matrix. It is necessary to point out that this phase will be continued to find a solution.
- Shaking: The aim of this phase is to make a shake in the existing solution. In this phase, several making neighbor structures are used and the neighborhood structure is changed.
- Local search: This phase aims to find the local optimal solution. After making the neighborhood in this phase, the local search method is applied to the changed solution. Swap, Insertion, and Reversion are used in this paper. The first operator (insertion) first selects two elements of i and j as a sample, and then take the i-th element from its place and locate it in the place of $j+1$ (one cell after j). The second operator (i.e., reversion) first selects the elements of i and j as a sample and then ordering of their arrangements, and all of the elements between them are reversed. The third move operator (i.e., swap) select two elements of i and j accidently and then change the place of their elements with each other.
- Stopping criterion: In each algorithm in the neighborhood, if in the number of repetitions *MaxIt2*, a better answer than the current answer will not be found, the neighborhood will be increased and if the algorithm in the neighborhood will be L_{max}, the neighborhood will be returned to l_1. The algorithm will be stopped if the number of repetitions reaches to *MaxIt*.

4.6. Hybrid SA and VNS Algorithm

The concept of the SA algorithm into VNS, it guarantees the effectiveness of VNS and alleviates the potential weakness of VNS (intensification) [42]. The reason is perhaps the strategies for searching the answer space by two algorithms. In the SA algorithm in each iteration, one of the methods for searching the neighborhood is used to produce a new answer. In this algorithm, there is not a specific procedure for using the methods for the searching neighborhood. However, in the VNS algorithm, first, the methods for neighborhood searching are arranged in terms of the extent of changes that applied in solution. Then, the algorithm starts from its search by the method of neighborhood searching with the least changes in answer, and if the solution does not find any consequent repetition, it will choose the next search that causes more change in the current solution. This causes the algorithm with a few changes to follow the better solution if the current solution is good. If a better solution cannot be found, the solution will be changed more or in order words, it will continue its search in more far space from the current solution. The pseudo-code of VNS-SA used in this paper is shown in (e.g., *Fig. 2*).

The SA algorithm uses temperature changes of the neighborhood structure. The VNS algorithm has no temperature, uses three defined neighborhoods, keeps a better solution in proportionate to that neighborhood, and omits the worse solution. However, the hybrid of SA and VNS (i.e., VNS-SA algorithm) starts with finding initial solutions. It continues in addition to conventional neighborhood structures of VNS, an internal loop algorithm out of neighborhood search in temperature until $l \leq l_{max}$. Then, a geometric function is used to reduce the temperature of SA *Eq. (31).*

$$T_{i+1} = \alpha T_i \,, \tag{31}$$

where α is temperature reducing parameter, T_i is the current temperature, and T_{i+1} is the new temperature. The algorithm will continue to reach to the determined final temperature.

```
Input: a set of neighborhood structures N_1, L = 1, 2, …, L_max
x = generate initial solution
T = T_0
While It < MaxIt
{
Until (it2 < MaxIt2)
        {
 Generate randomly Solution from neighborhood structures.
   if (xnew cost ≤ x.cost and xnew if feasible)
            x ← xnew
                else
                        Δ = xnew.cost − x.cost;
                        r = random();
                        p = exp(Δ / T);
                        if (r < p)
                        x ← xnew
                 end
   end
        }
           T = α.T_0
}
Return the best solution found.
```

Fig. 2. *Pseudo-code of the SA algorithm into VNS (i.e., VNS-SA).*

4.7. Taguchi Parameter Design

The results of the meta-heuristics are dependent on the values of the input parameters, it has been proposed. The parameters of the effect of the meta-heuristic algorithms on their suitable function, in which the Taguchi analysis is a statistical method used to adjust parameters. In this method, a rate named *S/N* is used to study solutions. At this rate, *S* is extant of utility, and *N* is a non-utility. As a result, the is to increase the quantity of this rate as possible [52].

From the standard table of the orthogonal arrays, the *L9* is selected as the fittest orthogonal array design, which fulfills all the minimum requirements. For each algorithm, the effect parameters are determined by trial and error. For each parameter, three levels are considered. Minitab Software is used to analyze the data. By considering a number of the selected factors and selected levels for the analysis, a suitable standard table is chosen for this study. Then, three typical problems are selected accidentally. Each problem by considering the given quantities for the parameters in each line of the Taguchi table to times runes. The mean objective function values are reported to Minitab17 software [47]. Finally, for each parameter, the level with more *S/N* as the best level of that parameter is considered.

4.8. Parameter Tuning

To solve the model, the parameters of the meta-heuristic algorithms are tuned according to (e.g., *Table 2*). In this table, T is the final temperature and α is the reduction multiple of temperature in the SA algorithm. T_0 is also the initial temperature. MaxIt and MaxIt2 respectively the maximum repeated number, the repetition number in each neighborhood, and the probable selection of neighborhood structure relating to the daily visit of customers in the VNS algorithm.

Table 2. *Parameters of the meta-heuristics algorithms.*

SA		VNS		VNS-SA	
Parameter	**Value**	**Parameter**	**Value**	**Parameter**	**Value**
T_0	100	*MaxIt*	300-500	T_0	100
α	0.98	*MaxIt2*	3	α	0.98
T	0.01			T	0.01
				MaxIt	300-500
				MaxIt2	10

5. Numerical Results and Comparison

The Solomon benchmark is a known benchmark used in VRPs. Solomon's problems are based on the features and characteristics that are classified. Each group is included in 8 to 12 problems and a maximum of customers is 100. These problems are included in three main groups (C, R, and RC), and two sub-groups (C_1, C_2, R_1, R_2, RC_1, RC_2). The problems are different in terms of four features. The quantity of demand, time windows, and service time. Also, the problems of a single depot are considered. There are two important points for using this benchmark in electric VRPs: First, this has not the possibility of specifying the place of charge stations. Second, the central depot is defined.

To produce the instance problems in small and large sizes, it is used as a combination of the Solomon benchmark and instances in the literature with a little change. The Euclidean distance of the cities from Solomon's benchmark and fuzzy time windows and time services of both points of pessimistic and optimistic values is then calculated. For each customer, a fixed quantity of pickup or delivery is considered. For each depot, the number of vehicles is allocated. The number of recharge stations candid place is specified in *([2+0.05C], [3+0.1C]*) interval.

The model for small-sized samples is solved by using of solving CPLEX in commercial software GAMS 24.1.2. The algorithms are running on the computer with specification Core i5, CPU 2.60 GHZ using software MatlabR2017a (64-bit) by Microsoft windows10 for small and large sizes.

5.1. Efficiency of the Algorithms of Small-Sized Problems

In small-sized problems, 10 problems are produced randomly and the results of an exact solution with GAMS software and CPLEX solver are compared to SA, VNS, and VNS-SA algorithms.

The results of a typical problem solving are shown in (e.g., *Tables 3* & *4*). C, D, E, K the data of the problem briefly characterize the number of customers, depots, charge stations, and the number of vehicles. Time at the table is expressed in terms of seconds and the error is expressed based on the following equation.

$$\mathrm{Gap\%} = \frac{\mathrm{RA} - \mathrm{RB}}{\mathrm{RB}} * 100. \tag{32}$$

RA and RB are solutions resulted from the algorithm and the best-resulted solutions. In this paper equation, as specified in the table, the quality of solutions resulted from three are suitable meta-heuristic algorithms. The average error rate of algorithms: SA, VNS, VNS-SA are at 0.45, 0.23, 0.12. respectively. The maximum error is SA as 0.92. In terms of time, GAMS is equal to 3.618.

Table 3. *Results of GAMS in small-sized problems.*

Instance	C-D-E-K	CPLEX	Time (s)
1	*4-2-1-3*	*194.56*	*0.1*
2	*5-2-1-3*	*293.96*	*0.3*
3	*5-2-1-4*	*304.2*	*0.2*
4	*5-2-2-4*	*342.54*	*0.27*
5	*6-2-2-4*	*312.15*	*0.38*
6	*7-2-2-4*	*367.51*	*4.81*
7	*7-2-3-4*	*325.73*	*4.14*
8	*8-2-2-3*	*547.12*	*8.45*
9	*8-2-3-3*	*603.64*	*8.2*
10	*9-2-2-4*	*714.6*	*9.33*
Average		*400.601*	*3.618*

Table 4. *Results of MATLAB in small-sized problems.*

Instance	C-D-E-K	SA	t(s)	Gap%	VNS	t(s)	Gap%	VNS-SA	t(s)	Gap%
1	*4-2-1-3*	*195.8*	*0.05*	*0.64*	*195.55*	*0.08*	*0.51*	*195.55*	*0.09*	*0.51*
2	*5-2-1-3*	*297*	*0.12*	*1.03*	*294.98*	*0.1*	*0.35*	*293.96*	*0.1*	*0.00*
3	*5-2-1-4*	*304.35*	*0.1*	*0.05*	*304.2*	*0.11*	*0.00*	*304.2*	*0.09*	*0.00*
4	*5-2-2-4*	*342.97*	*0.3*	*0.13*	*342.67*	*0.25*	*0.04*	*342.61*	*0.19*	*0.02*
5	*6-2-2-4*	*313.84*	*0.35*	*0.54*	*313.16*	*0.31*	*0.32*	*313.76*	*0.3*	*0.52*
6	*7-2-2-4*	*367.61*	*0.37*	*0.03*	*367.58*	*0.42*	*0.02*	*367.57*	*0.4*	*0.02*
7	*7-2-3-4*	*325.79*	*0.9*	*0.02*	*325.99*	*0.7*	*0.08*	*325.73*	*0.65*	*0.00*
8	*8-2-2-3*	*552.15*	*1.8*	*0.92*	*551.14*	*0.99*	*0.73*	*547.12*	*0.7*	*0.00*
9	*8-2-3-3*	*603.69*	*2.01*	*0.01*	*604.41*	*1.12*	*0.13*	*603.66*	*0.91*	*0.00*
10	*9-2-2-4*	*721.95*	*2.63*	*1.03*	*715.65*	*1.18*	*0.15*	*715.3*	*1.02*	*0.10*
Average		*402.51*	*0.86*	*0.44*	*401.53*	*0.53*	*0.23*	*400.94*	*0.45*	*0.12*

5.2. Efficiency of the Algorithms for Large-Sized Problems

In large-sized problems, groups similar to Solomon benchmark problems with the same dimension for 15-200 customer are chosen, and by considering the pickup or delivery of each customer. Then, a solving method is applied to them. The results of solving the typical problems are shown in (e.g., *Tables 5* & *6*) VNS-SA in large-sized instances is better than SA and VNS algorithms. The percent of error, the VNS-SA with 0.011 error has a very insignificant error. SA with average error 2.645 has a weaker instance than the proposed algorithm. The least time consuming is to solve the VNS-SA algorithm.

Table 5*. Results of SA and VNS for large-sized problems.*

Instance	C-D-E-K	SA	t(s)	Gap (%)	VNS	t(s)	Gap (%)
1	*25-2-4-6*	*33073*	*12.92*	*1.122*	*32870*	*11.20*	*0.501*
2	*25-2-5-6*	*34297*	*15.56*	*1.016*	*33952*	*14.32*	*0.000*
3	*30-3-5-7*	*42398*	*26.30*	*1.150*	*42199*	*22.85*	*0.675*
4	*35-3-6-7*	*43016*	*29.41*	*1.458*	*43019*	*35.30*	*1.465*
5	*40-3-6-8*	*51202*	*40.34*	*2.027*	*51043*	*47.60*	*1.710*
6	*45-4-5-8*	*59211*	*46.26*	*2.062*	*59159*	*56.20*	*1.972*
7	*50-4-6-9*	*62726*	*50.80*	*2.121*	*61423*	*93.56*	*0.000*
8	*50-4-7-9*	*77925*	*51.22*	*2.224*	*77840*	*96.21*	*2.112*
9	*60-5-8-10*	*86390*	*67.30*	*2.545*	*86108*	*109.20*	*2.210*
10	*60-5-9-10*	*108205*	*68.12*	*2.606*	*107795*	*123.65*	*2.217*
11	*70-6-10-11*	*118603*	*80.35*	*2.897*	*117862*	*191.41*	*2.254*
12	*80-6-11-11*	*125275*	*91.65*	*2.680*	*123815*	*250.00*	*1.484*
13	*90-7-11-12*	*140154*	*109.20*	*3.123*	*138078*	*273.22*	*1.595*
14	*100-9-12-14*	*150289*	*181.60*	*3.327*	*149795*	*319.65*	*2.987*
15	*100-10-13-15*	*167568*	*319.36*	*3.185*	*166288*	*321.30*	*2.397*
16	*120-14-15-18*	*186150*	*363.30*	*3.380*	*185220*	*412.60*	*2.863*
17	*130-14-16-19*	*187150*	*420.24*	*3.873*	*185720*	*435.60*	*3.079*
18	*160-15-17-20*	*213598*	*492.51*	*3.878*	*212101*	*455.32*	*3.150*
19	*180-15-20-22*	*214048*	*539.37*	*4.097*	*213101*	*490.14*	*3.637*
20	*200-16-22-27*	*239314*	*658.00*	*4.135*	*239117*	*525.25*	*4.049*
Average		*117030*	*183.19*	*2.645*	*116325*	*214.23*	*2.018*

According to the convergence rate of the proposed hybrid VNS-SA algorithm in (e.g., *Fig. 3*), it can result that as the problem size becomes large, the algorithm has better quality in terms of objective function values. The (e.g., *Fig. 4*) depicts the objective function values with different small-sized problems.

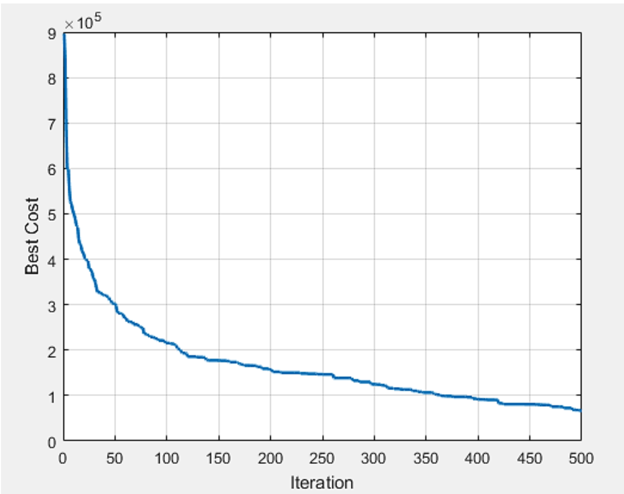

Fig. 3. *Convergence rate of the proposed hybrid VNS-SA algorithm.*

Table 6. *Results of VNS-SA for large-sized problems.*

Instance	C-D-E-K	VNS-SA	t(s)	Gap%
1	*25-2-4-6*	*32706*	*9.92*	*0.000*
2	*25-2-5-6*	*33975*	*11.76*	*0.068*
3	*30-3-5-7*	*41916*	*14.52*	*0.000*
4	*35-3-6-7*	*42398*	*21.03*	*0.000*
5	*40-3-6-8*	*50185*	*32.50*	*0.000*
6	*45-4-5-8*	*58015*	*45.99*	*0.000*
7	*50-4-6-9*	*61520*	*71.02*	*0.158*
8	*50-4-7-9*	*76230*	*73.15*	*0.000*
9	*60-5-8-10*	*84246*	*89.35*	*0.000*
10	*60-5-9-10*	*105457*	*92.50*	*0.000*
11	*70-6-10-11*	*115264*	*115.62*	*0.000*
12	*80-6-11-11*	*122005*	*125.48*	*0.000*
13	*90-7-11-12*	*135910*	*139.79*	*0.000*
14	*100-9-12-14*	*145450*	*150.36*	*0.000*
15	*100-10-13-15*	*162395*	*156.90*	*0.000*
16	*120-14-15-18*	*180064*	*173.10*	*0.000*
17	*130-14-16-19*	*180172*	*189.35*	*0.000*
18	*160-15-17-20*	*205623*	*232.70*	*0.000*
19	*180-15-20-22*	*205623*	*240.60*	*0.000*
20	*200-16-22-27*	*229811*	*273.09*	*0.000*
Average		*113448*	*112.94*	*0.011*

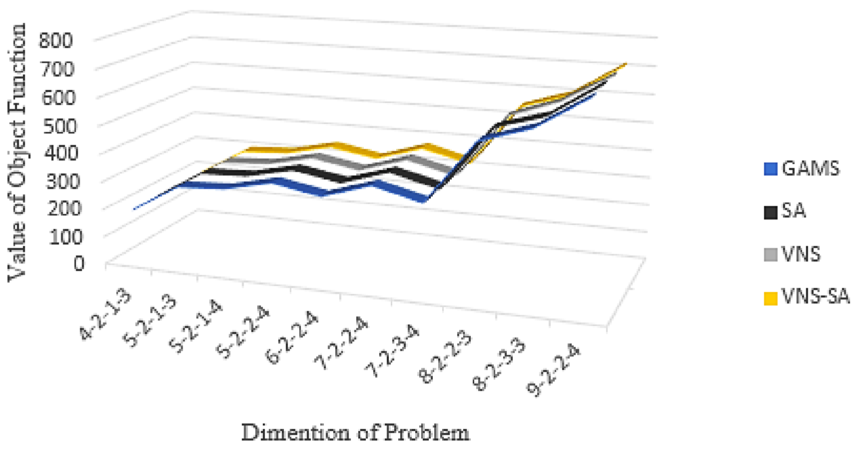

Fig. 4. *Objective function values for small-sized problems.*

6. Conclusions and Future Research

In this paper, a new mathematical model for an Electric Vehicle Routing Problem (EVRP) with fuzzy time windows was presented. The objective function of the presented model was to minimize the cost of the traveled distance and the penalty of time windows with a fuzzy approach. To show the efficiency of the proposed hybrid algorithm, the results were compared with the results obtained by the exact solution, Simulated Annealing (SA), and Variable Neighborhood Search (VNS) algorithms. Then, the proposed hybrid VNS-SA algorithm showed that its performance outperformed than the SA and VNS algorithms in small- and large-sizes problems. By considering the following issues that help to a deeper understanding of the effects of these factors on solutions, it is suggested to develop the model to make more practical the following future studies:

- Using a heterogeneous fleet and different speeds.
- Considering environmental factors (e.g., changing the temperature degrees), mountain routes, traffic, and queue in the battery recharge station.
- Considering the fuzzy stochastic and robust models.

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Steering Assisting with Path Detection and Car Detection

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PAPER INFO	ABSTRACT
Chronicle: *Received: 06 November 2020* *Reviewed: 12 January 2021* *Revised: 08 February 2021* *Accepted: 30 February 2021* **Keywords:** *Lane Detection.* *Car Detection.* *Steering Assistance System.* *Opencv.* *Numpy.* *Pycharm.* *Python.*	The recognition of pathways and identification of cars was seen with a prospective camera, which recognizes trajectories and predicts control points. The aim is to propose the location of the path. In this paper, lane detection algorithm Steering Assistance System (SAS) is introduced. Guiding helps to learn driving and anticipates the control points and defines the direction that makes it easy to learn in a potential way and a lane keeping assistance system which warns the driver on unintended lane departures. Path keeping is an important element for self-driving cars. This article describes the beginning to end adapting the approach to holding the car in the right direction.

1. Introduction

The enhancement of science and technology leads to make the life more comfortable than older days. The emerging technologies like neutrosophic shortest path [1]-[5], transportation problem [6]-[8], uncertainty problem [9]-[14], fuzzy shortest pathp [15]-[18], powershell [19], wireless sensor network [20]-[27], computer language [28] and [29], neural network [30], routing [31], image processing [32] making the products more intelligent and self-healing based. The smart city applications like smart water [33] and [34], smart grid, smart parking, smart resource management, etc. are based on IoT and IoE [35]- [38] technologies. This manuscript presents a beginning to end adapting approach to manage get the right guiding point to keep up the car in the way. A controlling help mechanical assembly of a car incorporates a path set unit to perceive a situation of ahead the car; a deviation assurance unit to decide if the car has a deviation propensity from a voyaging path; a guiding control unit to apply controlling power in a deviation evading bearing to the directing instrument when it is resolved that the car has a deviation inclination; a neighboring car identification unit to identify a neighboring car, which goes in front of the car in a path close to the voyaging path of the car; a methodology degree estimation unit to compute a methodology level of the neighboring car toward the car as shown in *Fig. 4*. In the guiding help framework [39] and [40], a path discovery sensor distinguishes path limit lines in forward looking pictures accepted by a camera and appraisals street boundaries. A methodical diagram on the

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improvement of picture arrangement investigation frameworks for street cars particularly, the driver's undertaking of keeping the path ought to be observed by a driver help framework. The primary driver that a driver is getting off the street are mindlessness and exhaustion [41].

In these frameworks, hearty and solid car recognition is the initial step. Car identification and following has numerous applications including platooning (i.e., cars going in rapid and shut down separation in parkways), unpredictable (cars going in low speeds and close separation in urban communities), and self-ruling driving as shown in *Figs. 2 (a)* and *(b)*. The street boundaries are as per the following: guiding bearing, ebb and flow, counterbalanced, melancholy edge of camera [42] and [43] (a positive worth methods upwards from the skyline).

2. Problem Definition

Distinguishing edge focuses and gathering them as lines is generally straightforward, yet the count cost for this cycle is exceptionally high. As a rule, this cycle is worked by application explicit equipment so as to fulfill time limitations. We intend to build up a hearty calculation to distinguish the path without application explicit equipment. The least demanding approach to identify the takeoff of the path is to check the Car's Present Position (CCP) in the path [44] and [45]. The position is assessed by the path identification calculation. For this reason, we disentangle the cycle used to gather identified edge focuses by anticipating them on to a street surface with low goal where straight lines are recognized. The subsequent issue is the trouble of choosing path limit lines on streets with complex path limits [39] and [46] in existing system as shown in *Figs. 1 (a)* and *(b).*

Fig. 1 (a). *Lane detection in existing system.*

Fig. 1 (b). *Lane with no curve in existing system.*

By observing certain activities of the driver, presumptions about his expectation can be taken. Clearly, setting the signal declares an after path change. Path takeoffs in the wake of setting the signal are without a doubt intended so the driver can move the controlling course and keep from mishap and learn safe driving.

3. Literature Review

Our point is to build up a path identification sensor that is adequately hearty to empower it to be applied to the SAS without the requirement for application explicit equipment. For this reason, we disentangle the cycle that gatherings distinguished edge focuses into lines, utilizing a street surface with low goal. Furthermore, we utilize a technique to choose the right path limit lines from numerous applicants utilizing design coordinating for scenes where path limits are intricate [43] and [46]. A technique and framework to recognize and distinguish an objective vehicle showing an exchanged or invalid tag incorporates a tag per user to per use the plate number conveyed by an objective vehicle in its region. A pair of spaced laser beams measures the speed and length of the traveling car.

3.1. Different Researcher's Contributions

Table 1. *A literature review of car and lane detection.*

Authors	**Years**	**Different Approaches to Solve Image Processing**
Risack et al.[42]	*2000*	*The author proposed incorporated our path keeping associate in a trial car and performed methodical analyses in genuine rush hour gridlock circumstances.*
Watanabe and Nishida [39]	*2005*	*The maker proposed to develop a way area sensor that is satisfactorily fiery to engage it to be applied to the SAS without the necessity for application unequivocal gear.*
Liu et al.[45]	*2006*	*The authors studied Charge-Coupled-Gadget (CCD) camera is utilized as the front-end detecting for this framework to distinguish the path marks and momentarily ascertain the horizontal deviation.*
Son and Mita [43]	*2009*	*The authors proposed car is partitioned into numerous important highlights through their appearances in preparing tests.*
Choudhury [47]	*2017*	*The author proposed a keen traffic observation framework, outfitted with electronic gadgets, works by speaking with moving cars about traffic conditions, screen rules and guidelines and dodge crash between cars.*

The past suggestion frameworks had certain holes in them:

- It is normally sheltered to accept that a driver is cognizant and watching the traffic when he slows down in *Fig. 3*.
- Lane, car location is difficult to distinguish for a renewed individual and hard to drive [48].
- The existing techniques gives great precision and effectiveness to top notch pictures however now and again give helpless outcomes to poor natural conditions like mist, fog, clamor, dust [45] therefore, this motivates us to provide a new model for society.
- Improve the precision of location and keeps from mishap.
- By checking certain activities of the driver, suspicions about his expectation can be taken [49].
- Helps to master driving least demanding and quickest way.

4. Description of the Research Work

By chance, it is expected that, without different cars drawing nearer from the back, a car begins switching to another lane to get in a neighboring path, and, after the car has started entering the adjoining path, other car is drawing closer from the back in the objective path [50]. For this situation, it might be appropriate that the car rapidly gets done with moving to another lane as opposed to quits moving to another lane to clear a path for the other car drawing nearer from the back. At the end of the day, the option to proceed differs as per conditions on the event, and there front, a driver may perhaps feel awkward if a path change is perpetually stifled at whatever point the methodology of the other car from the back is recognized [44] and [51].

A driving control gadget as indicated by the current development initiates a control for suppressing sideways development of a car towards a side item, when the side article present to the side of the car just as towards the back of the car is distinguished [43] and [52]. Additionally, the enactment of the control is suppressed regardless of whether the side item is distinguished, when without discovery of the side article, the car begins entering an adjoining path so as to move to another lane in *Table 2*.

4.1. Proposed Car and Lane Detection Method

Table 2. *Proposed detection method.*

Steps	Overview
Step 1	*Considering our method, we will need a video, later we need to train a model based on the features, shape of the car and draw lines o road which detects the lane.*
Step 2	*Here, we will create the car and lane detection python files for video.*
Step 3	*Here we will make use of open CV which will check all the video with the trained detector model.*
Step 4	*This last step includes displaying the car and lane in the video within the rectangular box.*

5. Result and Discussions

This part contains the exhibition correlation of the proposed calculation and existing calculations by taking distinctive execution boundaries and proposed calculations which give more precise outcomes than the current calculations. We assessed the presentation of our path recognizing sensor utilizing genuine street pictures. These pictures were taken both in the daytime and around evening time. On-street car location is trying to the point, that none of the strategies evaluated can unravel only it totally. Various techniques should be attempted and chosen dependent on the won conditions looked by the framework. Using even and vertical edges for HG is presumably the most encouraging, information based, approach revealed in the writing. Our involvement in utilizing edge data in reasonable tests has been extremely sure. This calculation has been executed on straightforward equipment utilizing two universally useful microcomputers.

Our proposed model can detect both lane and car and has better accuracy than the existing models. The proposed model detects the lane as shown in *Figs. 2 (a)* and *(b)*, and it can detect car as shown in *Fig. 3* and *Fig. 4* has the lane and car that have detected from the video.

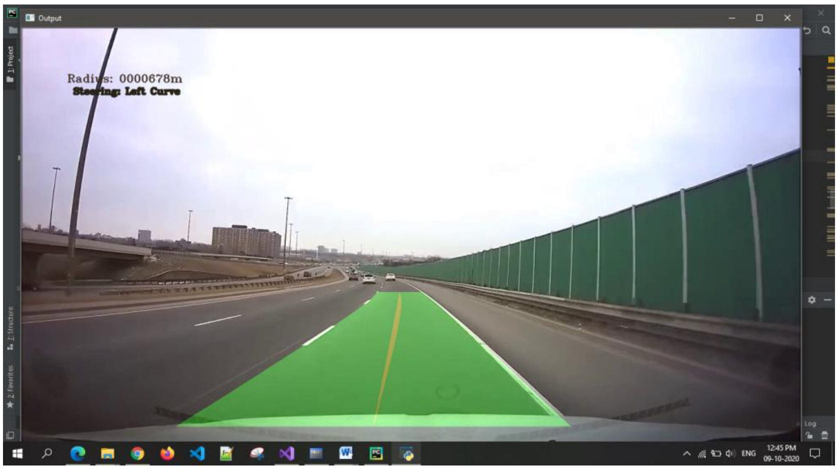

Fig. 2 (a). *Result showing lane detection in video using our proposed model.*

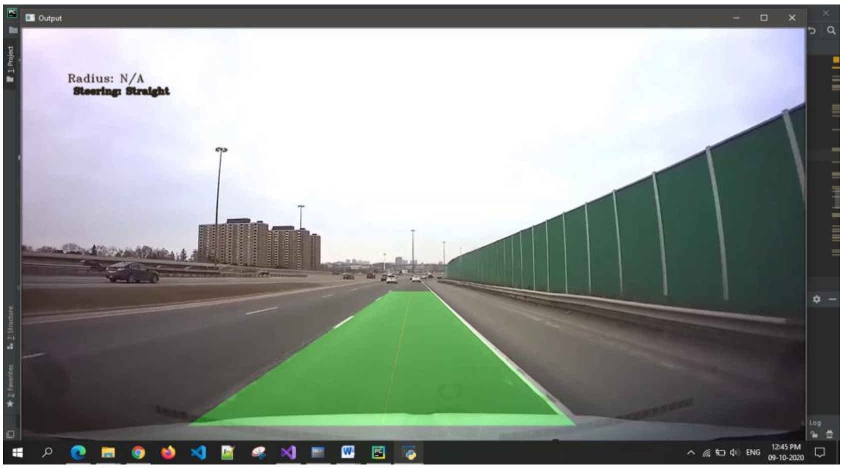

Fig. 2 (b). *Result showing lane detection in video using our proposed model.*

Fig. 3. *Result showing car detection.*

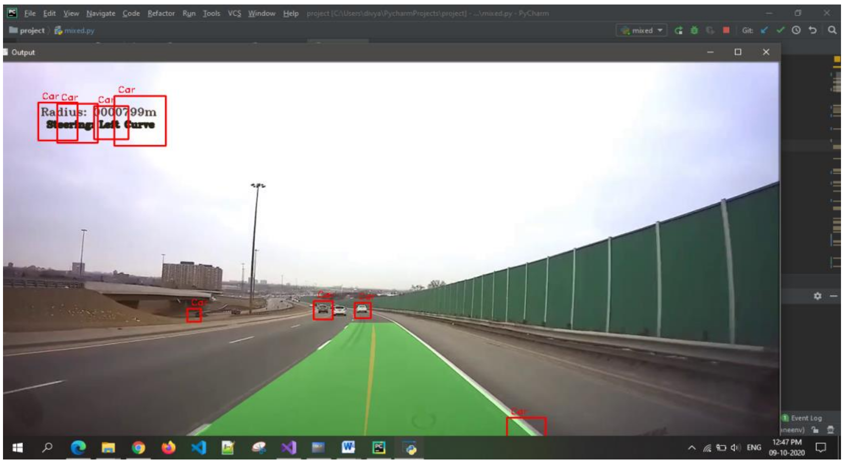

Fig. 4. *Result showing both lane and car detection using our proposed model*.

6. Conclusion

The path and cars recognition framework is getting famous progressively and savvy transportation frameworks for forestalling the vehicular incidents and improving the mishap ratio. The controlling help assists with distinguishing the position of tires. Some of them brought about erroneous outcomes in abnormal circumstances and the strategies grew so far are circumstances where clamor is least and natural conditions are likewise acceptable. Yet, they become mistaken or fizzle or not give proficient outcomes when there is any sort of commotion in the street pictures.

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Detection of an Imbalance Fault by Vibration Monitoring: Case of a Screw Compressor

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PAPER INFO	ABSTRACT
Chronicle: *Received: 02 December 2020* *Reviewed: 08 January 2020* *Revised: 21 February 2021* *Accepted: 06 March 2021* **Keywords:** *Vibration Analysis.* *Dynamic Behavior.* *Diagnostic.* *Fourier Transform*	The evolution of the means aimed at improving the availability of strategic equipment requires a credible level of maintenance with the development of original monitoring techniques such as vibration diagnostics. The vibration monitoring of strategic equipment through a vibration diagnosis requires mainly the identification of vibrational images of the different types of damage. However, one of the important vibration problems is essentially due to the phenomenon of imbalance. This phenomenon corresponds to an imbalance of the rotor due to the offset between the axis of inertia and the axis of rotation, which causes significant and cyclical vibrations. The aim of this study is to analyze the vibratory behavior of a screw compressor to improve its reliability and consequently its availability. To identify the imbalance fault, two sets of vibration measurements on April and January were fundamentally examined at the compressor level. Fourier transform based on spectral analysis was used to create a vibration detection approach with vibration signals. The comparison of the results obtained with that of the simulation resulting from the model of dynamic behavior of the compressor is conclusive.

1. Introduction

The detection of faults in rotating machines has been extensively studied by [1]-[3] and the performance analysis of industrial machines by modeling has been widely examined by [4] and [5]. The detection systems include experimental methods, the conventional methods one which therefore requires a high level of human expertise [1], [6]-[8] and combined techniques as Statistic Filter (SF) and Hilbert Transform (HT) which are joined with Moving-Peak-Hold method (M-PH) [9] and the unconventional one with an approach of artificial intelligence by the method of neural networks for the prediction of the vibratory behavior of mechanical systems [10] and [11].

For the detection of the imbalance defect, there is many techniques based on spectral analysis. In fact, frequency domain analysis is a technique widely adopted for the study of system vibrations [1], as it

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requires knowledge of the fundamental frequencies of the system [2]. Important to realize that other methods such as the numerical method allow to better identify the optimal location of the measurement points by a finite element code in order to model the vibratory behavior of the systems [12]. Recently, data-based methods have shown great potential in the intelligent diagnosis of defects [13] and [14]. A number of authors have investigated approaches based on artificial neural networks for prediction of misalignment defects and imbalance, this method is carried out in three main steps, initially a measurement step, then the learning step, and thirdly the prediction step [10]. Further, a new imbalanced fault diagnosis framework based on cluster Majority Weighted Minority Oversampling Technique (MWMOTE) and moth-flame optimization (LS-SVM) is proposed by Wei et al. [15] to address the fault diagnosis of rolling bearings under complex conditions. However, the detection of unbalance faults with frequency analysis has several advantages because it is a simple technique to implement, reliable and avoids the use of additional analysis software [1] and [2] adapt to industrial scale.

The commonly used tools to extract a fault indicator are spectral analysis techniques, several authors have used classical analysis techniques such as the Fourier transform [16] and [17], the Hilbert transform [18]-[20] or time-frequency distributions [21]-[23]. These aforementioned methods can identify unbalance faults thereby improving diagnostic performance.

Furthermore, spectral analysis makes it possible to detect anomalies through readings of acceleration or speed for certain frequency ranges in order to follow the state of degradation of an equipment. Similarly, it can also be used to carry out a diagnosis by interpretation of the shape of the vibratory signal [6] and [8]. It consists necessarily of filtering the measured vibration signal on the screw compressor and carrying out a systematic analysis to look for the presence of vibration images of all the faults likely to affect the installation in question. This allows access to the diagnosis successfully, that is to say, to identify precisely the nature of the anomaly and if possible specify its severity. In addition, the envelope spectrum is a tool particularly suitable for monitoring periodic shocks and it also makes it possible to quantify the impulse nature of the signal, hence the analysis of the envelope is best suited for detecting bearing faults [1], [6], [7] and [24].

Add to this, several phenomena of instability of the compressor are mainly the cause of their malfunctions and can create significant vibrations such as the imbalance and misalignment of the shafts, etc. As a matter of fact, the presence of small defects can lead to very significant vibration amplifications. Hence, the vibratory behavior of rotating machines and their processing is carried out on the measured signals, in many industrial applications, the vibration is characterized mainly by its frequency, its amplitude and its nature [25]. Eventually, imbalance is a term which characterizes a mass which is not perfectly distributed over a volume of revolution resulting in an imbalance. As the axis of inertia of the rotor no longer merges with the axis of rotation.

Even so, when the rotor is driven in rotation with an imbalanced mass, these results in centrifugal forces which increase the load on the bearings and a vibration appears on the rotor. It should be noted, that this is the specific characteristic of an imbalanced machine. This centrifugal force is thus the excitation force of the imbalance which is given by *Eq. (1)* [17] and [26]:

$$F_c = m\, r\, \omega^2. \tag{1}$$

On the other hand, the rotational speed is calculated using *Eq. (2)*:

$$\omega = 2\, \pi\, f. \tag{2}$$

With:

- F_c : imbalance strength (N);
- m : imbalance mass (kg);
- r : eccentricity (m);
- ω : rotational speed (rad/s);
- f : frequency of rotation (Hz).

The study presented consists of carrying out an analysis of the vibratory behavior of the screw compressor based on the operating parameters and the spectral indicators recorded.

The core objective of this work is to prevent failures in order to ensure optimal availability of the screw compressor, based on experimental tests, with regard to a vibration study.

2. Presentation of the Screw Compressor

Initially, the application system is a screw compressor driven by an electric motor, the two parts are coupled by means of an elastic coupling, the schematic diagram of the screw compressor with the kinematic chain and measuring points is clearly presented in *Fig. 1*.

The measurements were carried out precisely on the motor and the screw compressor, using a mobile accelerometer mounted on the measurement points indicated in *Fig. 1*.

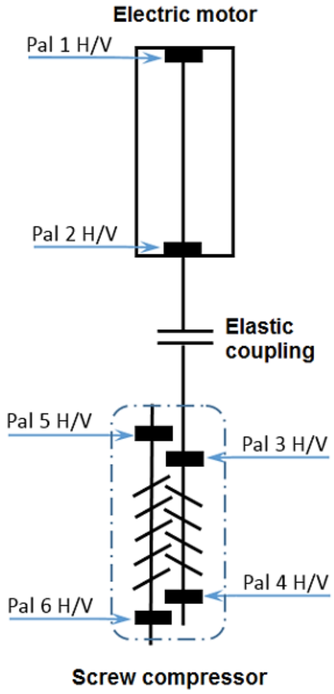

Fig. 1. *Schematic diagram of the screw compressor.*

Characteristics of the screw compressor:

- Electric motor:

  Power: 45 kW;

  Rotational speed: 2940 tr/min;

- Screw compressor:

  Power: 60 CV;

- Rolling:

  Pal1: single ball 6313;

  Pal2: single ball 6313;

  Pal3: cylindrical roller NU 211;

  Pal4: angular contact 7311B;

  Pal5: cylindrical roller NU 211;

  Pal6: angular contact 7311B;

- Coupling:

  Elastic;

- Vibration measurement point:

  H: horizontal;

  V: vertical;

- Fixing:

  Flexible supports;

Five series of measurements are respectively collected over the last two years of monitoring the evolution of the overall level of the six compressor stages are summarized in *Table 1*.

To begin with, the evolution of the overall level allows us to understand the state of the compressor by comparing the values of 'NG' with ISO standard 10816, but it does not allow to identify and discover the real causes behind the failure.

We only used the two series of vibration measurements of January 15 and April 20 in global 'NG' level in *Table 1* to carry out this study.

Table 1. *Measurement of the last two years reference of vibration (NG).*

Measuring point	July 29 RMS (mm/s)	August 21 RMS (mm/s)	February 8 RMS (mm/s)	April 20 RMS (mm/s)	January 15 RMS (mm/s)
Rh Pal1	*2.46*	*2.81*	*2.63*	*2.71*	*9.21*
Rv Pal1	*2.23*	*2.46*	*2.33*	*4.94*	*5.14*
Rh Pal2	*2.1*	*2.62*	*2.82*	*3.1*	*9.01*
Rv Pal2	*3.27*	*3.39*	*3.68*	*4.17*	*5.14*
Rh Pal3	*5.22*	*5.72*	*5.23*	*5.14*	*7.43*
Rv Pal3	*5.03*	*5.23*	*6.59*	*7.85*	*7.85*
Rh Pal4	*6.14*	*6.34*	*6.05*	*6.88*	*6.87*
Rv Pal4	*5.11*	*5.33*	*5.82*	*7.36*	*7.43*
Rh Pal5	*6.21*	*6.4*	*6.01*	*4.36*	*1.94*
Rv Pal5	*5.85*	*6.3*	*6.69*	*8.14*	*6.10*
Rh Pal6	*6.34*	*6.95*	*6.76*	*6.98*	*7.53*
Rv Pal6	*4.65*	*4.75*	*4.56*	*4.85*	*6.01*

3. Vibratory Behavior

3.1. Dynamic Behavior Modeling

The representation of *Fig. 2* is a simplification of the manifestation of the imbalance. Several damages cannot be located precisely as being a mass *(m)* and also fixed at a specific place. Indeed, for example, corrosion, wear or a deposit of impurities are distributed more or less uniformly over the entire surface, which thus has the effect of shifting the center of gravity. Add to this, the imbalance is modeled by a point mass placed at a given distance *(r)* from the axis of rotation *(Oy)* and a given distance *(d)* in accordance to the origin of the reference *(O)* as shown in *Fig. 2* and *Fig. 3*.

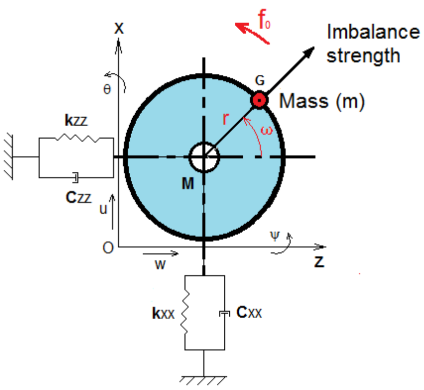

Fig. 2. *Imbalance modeling.*

The fan of *Fig. 2* is modeled by a node having four degrees of freedom: two rotation *(ψ)* and *(θ)* around the axes *(Ox)* and *(Oz)* and two displacements of translation *(u)* and *(w)* along the axes *(Ox)* and *(Oz)*. The disc is considered to be rigid and symmetrical.

The coordinates of the imbalance in the fixed coordinate system *R (Oxyz)* are given by:

$$\overrightarrow{OG}_{|R} = \begin{Bmatrix} u + r\cos\omega t \\ Cte \\ w + r\sin\omega t \end{Bmatrix}. \quad (3)$$

Hence the speed:

$$\overrightarrow{V_{G/R}} = \frac{d\overrightarrow{OG}}{dt} = \begin{Bmatrix} \dot{u} + r\,\omega \sin \omega t \\ 0 \\ \dot{w} - r\,\omega \cos \omega t \end{Bmatrix}. \tag{4}$$

The kinetic energy of the imbalance:

$$2\,T = m\left\|\overrightarrow{V_{G/R}}\right\|^2. \tag{5}$$

$$2\,T = m(\dot{u}^2 + \dot{w}^2 + \omega^2 r^2 + 2\omega r\,\dot{u} \sin \omega t - 2\omega r\,\dot{w} \cos \omega t). \tag{6}$$

The term *($\omega^2 r^2 / 2$)* is constant and will not occur in the equations, the expression for kinetic energy can be jointly approximated by:

$$T = m\omega r(\dot{u} \sin \omega t - \dot{w} \cos \omega t). \tag{7}$$

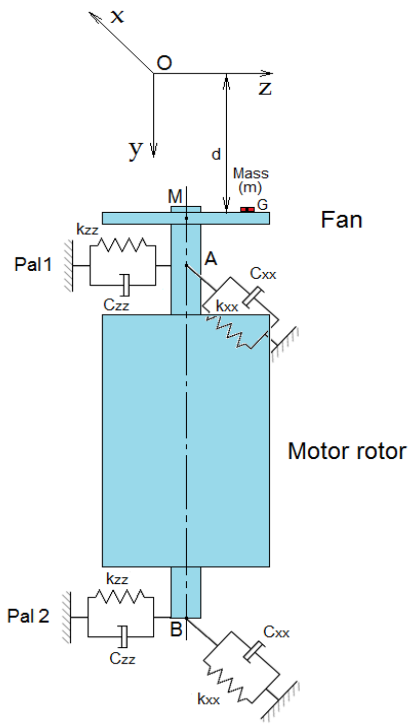

Fig. 3. *Simple model of the motor shaft line.*

In *Eq. (1)*, the term *(ω^2)* indicates that the force increases with the square of the speed. This explains why compressors at high speed 2940 rpm are very sensitive to the imbalance fault. Therefore, his force *(F)* rotates at the same frequency as the rotor and it consequently produces sinusoidal type vibrations on the bearings which rotate at the same frequency as the rotation frequency *(f)*.

$$F = F_C \sin(2\pi f t). \tag{8}$$

Additionally, the imbalance defect, perhaps either out of mechanical origin in the form of degradation or deposits of impurities on the rotor or is of thermal origin caused by a modification of the geometry of the rotating parts due to the rise in the temperature. In most cases, this type of fault can be remedied either by cleaning or by balancing. In this modeling, the rotating system consists of a motor rotor, a fan similar to a disc and two bearings *(Pal1)* and (*Pal2)* which are subjected to imbalance forces. However, the behavior equations will be deduced from a formulation of the application of the lagrange equations.

In the case of generalized coordinates, the lagrange equations are written as the following:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial U}{\partial q_i} = Q_i. \tag{9}$$

Where *(T)* and *(U)* are the kinetic and potential energies respectively. Correspondingly, the *(q_i)* are the generalized coordinates of the system and the *(Q_i)* are the generalized forces. Henceforth, the objective from these lagrange equations of this modeling that we can deduce the equation of motion in order to identify the response *x (t)*.

The Lagrange equation *Eq.(9)* applied to kinetic energy:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} = -m\,\omega^2 r \begin{Bmatrix} \cos \omega t \\ 0 \\ \sin \omega t \end{Bmatrix}. \tag{10}$$

After all, imbalances are the forces of asynchronous excitations. Certainly, the bearings are modeled in *Fig. 3* by considering the shaft as a linear viscoelastic solid, it is about the model of Kelvin-Voigt, characterized by a damping *(C_{ii})* in parallel according to *(Ox)* and *(Oz)* and a stiffness *(k_{ii})*.

3.2. Signal Processing

In the field of signal processing, we are particularly interested in the sinusoids of the *x (t)* response.

The displacement signal *x (t)* of the screw compressor is given by *Eq. (11)* [17]:

Displacement:

$$x(t) = A \sin(2\pi f\, t + \varphi). \tag{11}$$

Velocity:

$$v(t) = 2\pi f A \cos(2\pi f\, t + \varphi). \tag{12}$$

Acceleration:

$$a(t) = -(2\pi f)^2 A \sin(2\pi f\, t + \varphi). \tag{13}$$

With:

- $A \geq 0$: amplitude of the sinusoid in the signal unit;
- φ : phase of the sinusoid in radian;
- f : the frequency of the sinusoid in Hertz.

In these equations *Eq. (12)* and *Eq. (13)*, the speed increases in proportion to the frequency *(f)*, while the acceleration increases with the square of the frequency, which apply only to sinusoidal vibrations.

In this work, we present the application of the spectral analysis method to the diagnosis of the screw compressor, based on the Fourier transform. We consider that the signal *x (t)* is absolutely integrable at *] -∞, + ∞ [*. The Fourier transform of this signal is equal to [27]:

$$x(w) = F[x(t)] = \int_{-\infty}^{+\infty} x(t)e^{-j\omega t}\, dt. \tag{14}$$

The search for damage, the right monitoring of their evolution and the diagnosis of the state of the compressor are possible. By all means, if one knows the vibratory symptoms associated with each defect likely to affect this compressor, that is to say, if we know the vibrational images induced by these defects. As a result, for the defect of the imbalance subject of this study it clearly manifests itself by [7]:

– An increase in the overall level chosen at low frequencies V_{eff} [10-1000 Hz].
– A clear increase in the amplitude of the fundamental frequency (f_0):

$$f_0 = \frac{N}{60}. \tag{15}$$

With:

– N: rotational speed en tr/min;
– f_0: frequency of rotation en tr/s [Hz].

As illustrated above, the vibrations are the result of forces, which can be of different origins, they are transmitted to the structure via the bearings and accordingly to the foundations via the fixings. The measurements were carried out on the bearings placed on the machine structure, using a quartz 'ICP' accelerometer type sensor with a sensitivity of 100-mV / g. From the sent signals from the sensor, the spectra are designed for the two series of experimental measurements on January 15 and April 20.

In this context, and to theoretically simulate the effect of the imbalance of the screw compressor we used the software (Matlab), the *Fig. 4 (a)* represent the response *x (t)* and the *Fig. 4 (b)* represent the response *v (t)* of the designed imbalance model. The Fourier transform of the speed signal is represented by the *Fig. 4 (c)*.

The two time-based signals *x (t)* and *v (t)* obtained by (Matlab) have a frequency *f0 = 49 Hz* which corresponds to the speed of rotation of the shaft 2940 rpm acquired during the duration of one second

The Fourier transform allows the passage from the time domain to the frequency domain. The Fourier spectrum of *Fig. 4 (c)* of the vibratory time signal *x (t)* is broken down into frequency components. Thus, this Fourier transformation makes it possible to know the spectral energy or power content present in the localized signal. We can clearly see a high amplitude peak at the frequency 49 Hz which corresponds to the frequency of rotation of the screw compressor.

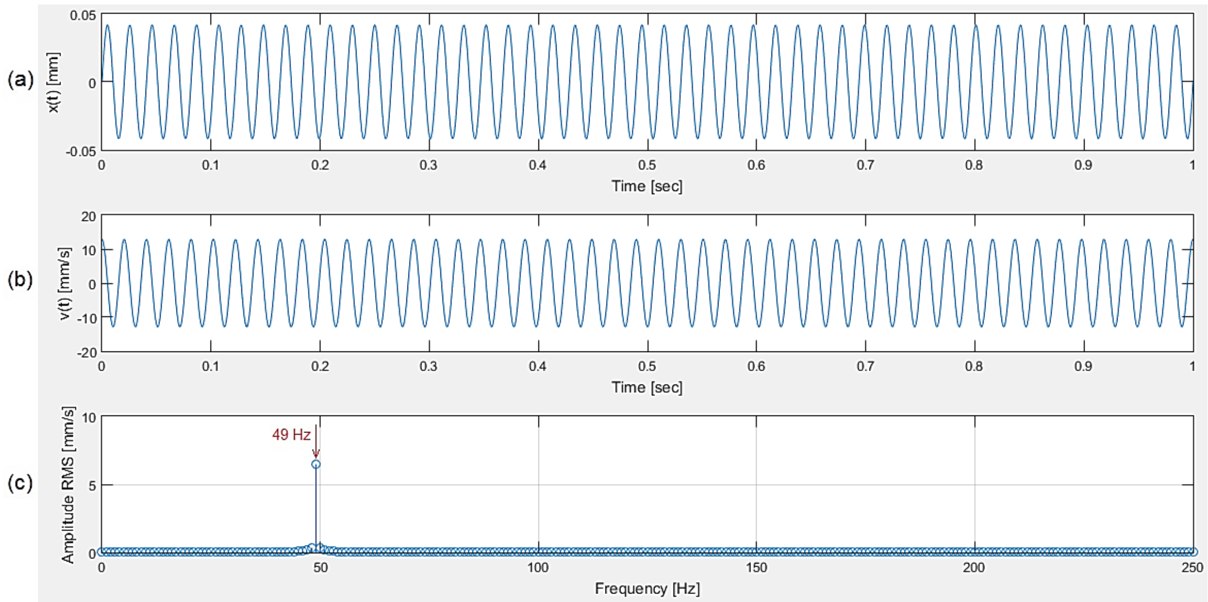

Fig. 4. *Theoretical simulation of the imbalance effect ((a) time curve of displacement, (b) time curve of velocity, and (c) theoretical spectrum of RMS velocity).*

4. Results and Discussion

From two series of experimental measurements of the spectra measured on January 15 and April 20, we can easily distinguish in *Fig. 5 (a)* high amplitude component peak at the frequency 49 Hz.

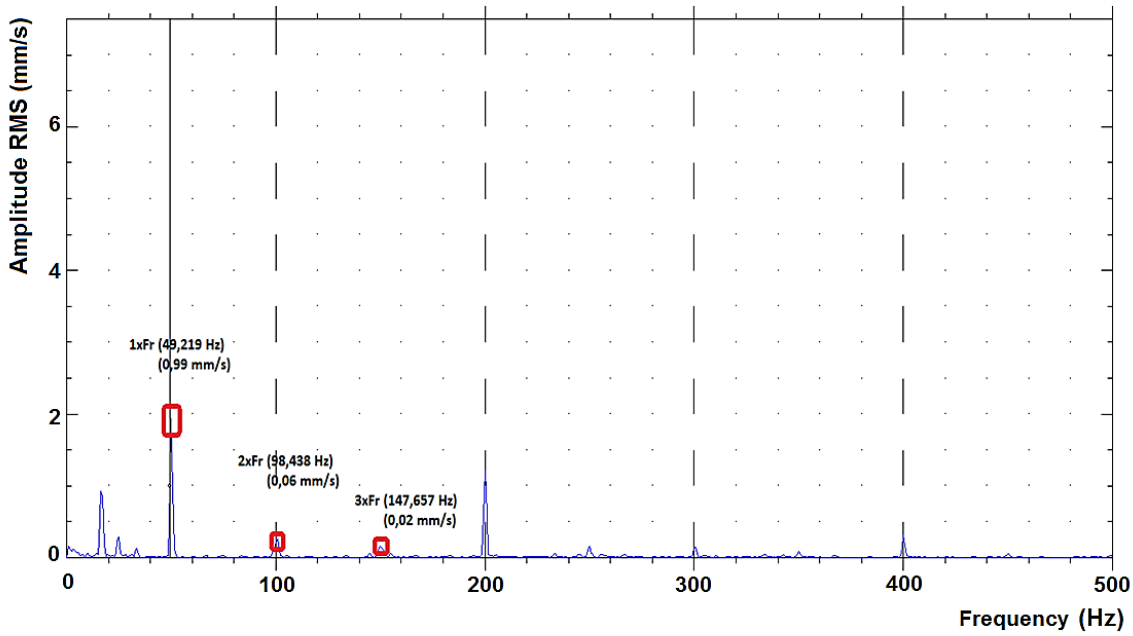

Fig. 5. *Pal 1 H spectrum recorded on April 20.*

As has been noted, the spectrum of *Fig. 5* shows a weak imbalance of 0.99 mm/s at the frequency of rotation $f_0 = 49\ Hz$ with an overall level of vibration 'NG' which is equal to 2.71 mm/s. The frequency signal of an imbalance fault looks like *Fig. 4 (c)* in the simulation.

The spectrum of *Fig. 6* of the date January 15 shows a marked increase at the amplitude of the fundamental frequency of 6.88 mm/s with an increase in the overall level 'NG' of vibration which is equal to 9.21 mm/s.

We compare in *Fig. 7* the evolution of the two frequency signals that of *Fig. 5* and that of *Fig. 6* measured at the point (Pal1H). We can see the presence of a major defect. The results obtained from the spectral analysis show that the amplitude at the frequency of rotation $f_0 = 49\ Hz$ of the reference signal in *Fig. 5* has increased by 7 times compared to that of *Fig. 6*, which implies an effect of imbalance. At the same time, when we are in a case of imbalance, there is always a large peak which corresponds to the frequency $(1 \times f_0)$ in the spectrum. In addition, we can see that only this peak is visible and relevant. An imbalance will therefore induce, in a radial plane, a vibration, the spectrum of which has an amplitude component predominant at the frequency of rotation of the rotor.

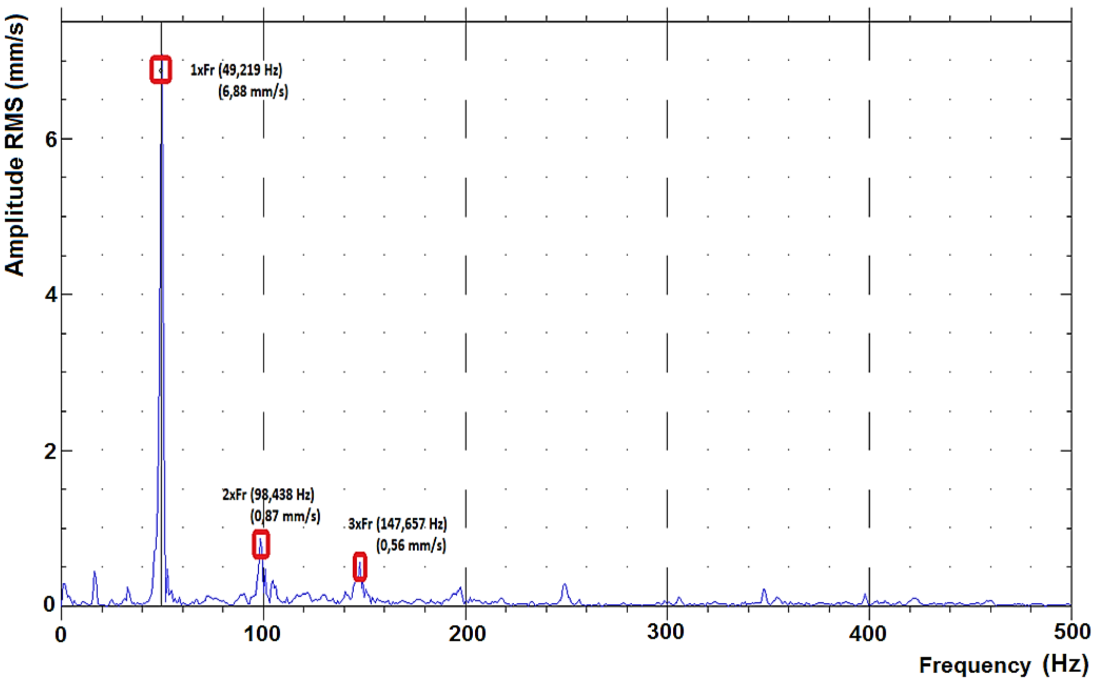

Fig. 6. *Pal 1 H spectrum recorded on January 15.*

All things considered, for a given equilibrium class, the overall level of induced vibration is compared with the thresholds set in standard ISO 10816 [7]. This makes it possible to judge the acceptability of the imbalance and the need or not to proceed with a balancing. In this case, the screw compressor motor rotates at 2940 rpm and has a power of 45 kW with a fixing made with flexible supports, the ISO 10816 standard imposes a danger threshold of 7.1 mm/s while the measurement carried out greatly exceeds this value 9.21 mm/s. This problem is corrected by an on-site balancing after dismantling and maintenance in the workshop.

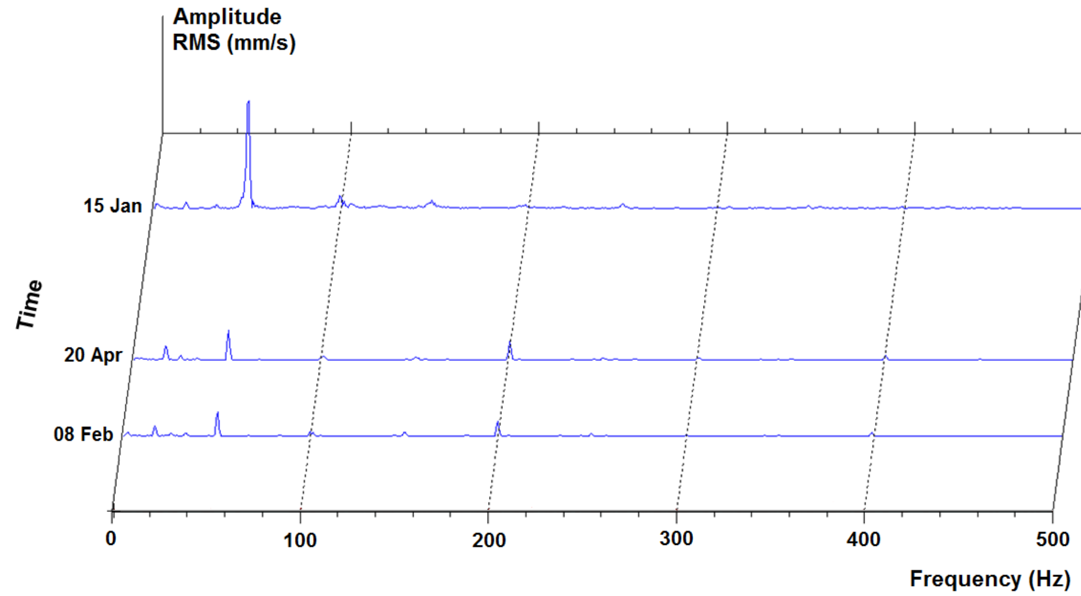

Fig. 7. *Comparison of the spectra of the Pal 1 H point of April 20 and January 15*.

As has been noted, that spectral analysis based on the Fourier transform present a technique that is simple to implement for industrial applications. On the other hand, if a precise description of the vibratory behavior is required, consequently theoretical analyzes of the dynamic behavior by additional analysis software are preferred. In reality, the choice of the characteristics and parameters of the system has a direct impact on the type of fault identification, since the estimation of the imbalance fault is not simple in the time domain. However, the identification of the frequency domain makes it possible to directly and effectively identify this defect.

5. Conclusion

In brief, monitoring the vibrations of this screw compressor plays a pivotal role in improving the reliability of this system. Nevertheless, mastering the vibration problems addressed in practice on this type of equipment increases their performance. To recap what has been analyzed so far in this article, the modeling of the vibrations of a screw compressor was examined, the detection and analysis of the vibrations were carried out using spectral analysis technique, to develop specific spectra using signal-processing techniques. This proposed approach was based on the modeling of the defects affecting the screw compressor, it make it possible to precisely identify the imbalance fault and to specify that it is very important in order to develop an accurate, correction of this problem.

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Robust Multi-Objective Hybrid Flow Shop Scheduling

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ABSTRACT

Scheduling is an important decision-making process that aims to allocate limited resources to the jobs in a production process. Among scheduling problems, Hybrid Flow Shop (HFS) scheduling has good adaptability with most real world applications including innumerable cases of uncertainty of parameters that would influence jobs processing when the schedule is executed. Thus a suitable scheduling model should take parameters uncertainty into account. The present study develops a multi-objective Robust Mixed-Integer Linear Programming (RMILP) model to accommodate the problem with the real-world conditions in which due date and processing time are assumed uncertain. The developed model is able to assign a set of jobs to available machines in order to obtain the best trade-off between two objectives including total tardiness and makespan under uncertain parameters. Fuzzy Goal Programming (FGP) is applied to solve this multi objective problem. Finally, to study and validate the efficiency of the developed RMILP model, some instances of different size are generated and solved using CPLEX solver of GAMS software under different uncertainty levels. Experimental results show that the developed model can find a solution to show the least modifications against uncertainty in processing time and due date in an HFS problem.

1. Introduction

Scheduling is an important task in production systems that aims to optimize one or more objectives under activity constraints and limited resources [1]. Its applications are ranging from production and manufacturing to transportation and logistics systems [2]. In fact, the purpose of production scheduling is to use various kinds of resources in an optimal way in a time based schedule. Development of an efficient scheduling method can results in productivity improvement of an organization [3]. One of the scheduling environment that is adaptable with most real world industry problems is the Hybrid Flow Shop (HFS) which is a difficult problem to solve [4]. A set of jobs flow through multiple stages in the

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same machine order in a HFS problem. There are several parallel machines at each stage, which can be identical, uniform and unrelated. Jobs have to be processed by one of the machines at each stage, and the flow of jobs through the shop is unidirectional [5] and [6]. There are several research dealt with the HFS with identical machines [7]-[9]. Besides, the main objective functions of the reviewed HFS models are the maximum value, the average, the sum or the weighted sum of the average, the sum or the weighted sum of completion time of jobs [10]-[13], delays or tardiness of jobs [4] and [9], and flow time [14] and [15].

To model the HFS problem, there are mainly six methods. Two methods are position-based approach, three methods are based on the precedence relation between operations and one is based on discrete time periods. The position-based methods assume that each machine (Wanger's modelling [16]) or each stage (Naderi's modelling (i) [12]) has several sections regarded as a processing position according to the time sequence. A schedule plan should arrange jobs at each processing position. Precedence relation-based methods consider: i) precedence relation between two adjacent or non-adjacent operations on the same machine (Manne's modelling [17]), ii) only two adjacent operations (Guinet et al. [18]) on the same machine, and iii) precedence relation of all jobs at each stage (Naderi's modelling (ii) [12]). The method based on the discrete periods (Bowman's modelling) introduces binary decision variables. These variables determine whether an operation processes on one machine in a certain period. Obviously, the large number of binary decision variables makes solving the problem harder [19]. Recently, Meng et al. [20] further have studied the above mentioned models for the HFS problem where their objectives were to minimize the makespan. Based on the experimental results, they have shown that all six existing models are correct and only are formulated based on different ideas.

The HFS problem under uncertainty has received much attention in the recent years [9], [10], [13], [15], [21], and [22]. Since real scheduling problems might be affected by various sources of uncertainties, ignoring them may lead to poor schedules [2] and [23]. Therefore, scholars have introduced different methods where uncertainty is directly taken into account. Some methods consider the random variables as input and some of them are worst-case approaches in which uncertain parameters belong to uncertainty sets. These frameworks are called respectively stochastic programming and Robust Optimization (RO). Due to hedging against adverse conditions of a system and being more insensitive against the future fluctuations of parameters, robust schedules are desirable from a practical perspective [2].

In the sequel, first we review some robust optimization related research. Li and Ierapetrito [24] addressed uncertainty in scheduling problem and studied three robust counterpart optimization formulations. They concluded that the most appropriate model to deal with uncertainty is the formulation proposed by Bertsimas and Sim (B&S) [37]. Rahmani and Heydari [14] studied the flow shop scheduling under unexpected arrivals of the new jobs and uncertain processing times. They proposed a new approach in which an initial robust solution was determined proactively against uncertain processing times at first. Then, an efficient reactive method was produced to deal with unexpected disruption. Based on their computational results, the proposed method could produce better solutions in comparison with the classical heuristic approaches. Nagasawa et al. [25] proposed a robust schedule to limit the peak power consumption by considering unexpected fluctuation in the processing time. They performed simulations in order to find the optimal amount of idle time, which leads to having longer makespan and decreasing the production efficiency. They also proposed a robust scheduling model that considered random processing times and the peak power consumption. Based on the results of experiments, the performance of the schedule produced by the proposed method was superior to the initial schedule and to a schedule produced by another method. Thus, they concluded that the use of random processing times could limit the peak power. Shahnaghi et al. [26] provided a robust model for flow shop with batch processing by using B&S robust optimization approach in which the processing times and the size of jobs are uncertain. Based on the PSO algorithm results, B&S approach has better performance compared to the Ben-Tal model. Emami et al. [27] presented a robust optimization approach for the order acceptance scheduling problem with non-identical parallel machines based on the B&S approach. Job profits and the processing times considered random parameters. Hamaz et al. [28] applied the B&S approach to formulate the scheduling problem with uncertain processing times as a two-stage robust optimization problem. Ding et al. [29] studied the problem of program performance scheduling with accepting strategy under uncertainty of actual situation. This paper built a min-max robust optimization model based on B&S approach to minimize the performance cost and determine the sequence of the programs. Jamili [30] intended to deal with a new variant of job shop scheduling problem under uncertain processing times. In this research, a robust model was proposed by applying the B&S approach. Next, some algorithms were employed to solve the practical cases that were demonstrated as a proof of the effectiveness of the new approach. Bougeret et al. [2] studied the scheduling with budgeted uncertainty on a single machine that minimizes the makespan on parallel and unrelated machines. The processing times could take any value in the budgeted uncertainty set introduced by B&S [2]. Goli et al. [31] proposed a novel Robust Mixed-Integer Linear Programming (RMILP) model for the flow shop scheduling problem with outsourcing option based on the B&S approach, which uncertain processing time is one of their assumption. The comparison analysis demonstrated the superiority of the proposed model against the previous non-linear programming model in the literature [31]. *Table 1* also summarizes these researches.

Table 1. *Studies on the applying B&S approach to various scheduling problems.*

year	Refrence	Problem	Uncertain Parameters
2019	*Bougeret et al. [2]*	*Single machine scheduling.*	*Processing time.*
2019	*Goli et al. [31]*	*Flow shop scheduling.*	*Processing time.*
2019	*Jamili [30]*	*Job shop scheduling.*	*Processing time.*
2019	*Hamaz et al. [28]*	*Basic cyclic scheduling*	*Processing time.*
2018	*Ding et al. [29]*	*Program performance scheduling.*	*Actual situation strategy.*
2017	*Emami et al. [27]*	*Order acceptance scheduling.*	*Job profits and the processing times.*
2016	*Shahnaghi et al. [26]*	*Flow shop.*	*Processing times and the size of jobs.*

The present study applies the B&S robust optimization approach to an extended version of the HFS problem of Meng et al. [20] under uncertainty. This extended model considers the identical parallel machines and includes two uncertain parameters: processing time P_{ik} and due date d_i. Besides, we consider the total tardiness of jobs along with the makespan as a second objective function.

The remaining part of this paper is organized as follows: Section 2 briefly discusses B&S robust optimization approach. In Section 3, the HFS model and its robust counterpart formulation are discussed. Since the proposed problem is a multi-objective one, Section 4 introduces the applied fuzzy goal programming approach for solving the MILP model. Section 5 presents the computational results, model validation and sensitivity analysis. Finally, Section 6 describes the conclusions and future research directions.

2. B&S Robust Optimization Approach

One of the widely used and reliable approaches for dealing with uncertainty is the so-called RO. It has been applied in different real world applications, see for example [31]- [33]. This approach is capable to obtain a solution close to the optimal solution that is guaranteed to be feasible and good for all or most possible realizations of the uncertain parameters [34]. Applying RO to scheduling planning cause to generate more reliable schedule to minimize the effect of disruptions when the schedule is implemented [10]. Soyster [35] was the first who formulated robust linear optimization, but his model was too conservative [35]. Ben-Tal and Nemirovski [36] have introduced an efficient algorithm for modeling uncertainty of data based on ellipsoidal uncertainty sets. Being a conic quadratic problem, this formulation cannot be suitable for optimizing discrete problems. Bertsimas and Sim [37] have developed a different robust formulation that is linear, capable to control the level of conservatism and readily extends to discrete optimization problems. Based on the robust formulation of [37], we propose a robust HFS model under uncertain processing time and due date. In the sequel, first we review Bertsimas and Sim approach.

Consider the linear optimization problem

$$\begin{aligned} &\min_{x} \sum_{j\in J} \widetilde{a_{0j}} x_j, \\ &\text{s.t.} \\ &\sum_{j\in J} \widetilde{a_{ij}} x_j \le b_i \qquad \forall\, i \in I, \\ &l_j \le x_j \le u_j \qquad \forall\, j \in J. \end{aligned} \tag{1}$$

Let J denotes the set of coefficients $a_{ij},\ j \in J$ that are associated with parameter uncertainty, i.e. $j \in J$ takes values according to a symmetric distribution with mean equal to the nominal value $\overline{a_{ij}}$ in interval$[\ \overline{a_{ij}} - \widehat{a_{ij}}\ , \overline{a_{ij}} + \widehat{a_{ij}}\ ]$. The $\widehat{a_{ij}}$ is known as perturbation of uncertain parameter. A parameter Γ that takes values in the interval $[0, |J|]$ is introduced to control the level of conservatism of the solution. The main idea is that only a subset of the coefficients $a_{ij},\ j \in J$ will usually change. Therefore, being protected against the fluctuations, B&S approach considers that $\lfloor \Gamma \rfloor$ of these coefficients are allowed to change, and one of the coefficient a_{it} changes by$(\ \Gamma - \lfloor \Gamma \rfloor)\widehat{a_{it}}$. Based on this approach, the robust solution will be feasible if less than Γ uncertain coefficients change. Furthermore, there is high probability that the obtained robust solution will be feasible even if more than Γ change [37]. Based on this, the proposed Robust Linear Programming is developed as follows:

$$\begin{aligned} &\min t, \\ &\sum_{j\in J} \overline{a_{0j}} x_j + \max_{\{S\cup\{t\}|S\subseteq J, |S|=\lfloor\Gamma\rfloor, t\in S\setminus J\}} \left\{ \sum_{j\in S} \widehat{a_{0j}} |x_j| + (\ \Gamma - \lfloor\Gamma\rfloor) \widehat{a_{0t}} |x_t| \right\} \le t, \\ &\sum_{j\in J} \overline{a_{ij}} x_j + \max_{\{S\cup\{t\}|S\subseteq J, |S|=\lfloor\Gamma\rfloor, t\in S\setminus J\}} \left\{ \sum_{j\in S} \widehat{a_{ij}} |x_j| + (\ \Gamma - \lfloor\Gamma\rfloor) \widehat{a_{it}} |x_t| \right\} \le b_i\ \forall\, i \in I. \end{aligned} \tag{2}$$

Based on the *Model (2)*, if Γ is an integer, the constraint is protected by $\beta(x,\Gamma) = \max_{\{S\cup\{t\}|S\subseteq J, |S|=\lfloor\Gamma\rfloor, t\in S\setminus J\}} \{\sum_{j\in S} \widehat{a_{ij}} |x_j|\}$; when $\Gamma = 0$, the constraints turn to that of the nominal problem; if $\Gamma = |J|$, the *Model (2)* turns to Soyster's method because it means that all of the coefficient a_{ij} will change. Hence, the robustness of the approach against the level of solution conservatism can be adjusted by varying $\Gamma \in [0, |J|]$. *Model (2)* in the current form is nonlinear. To reformulate it as a linear model, the conservation function of i^{th} constraint is calculated for the given vector x^*:

$$\beta(x^*.\Gamma) = \max_{\{S\cup\{t\}|S\subseteq J,|S|=\lfloor\Gamma\rfloor,t\in S\setminus J\}} \left\{\sum_{j\in S} \widehat{a_{ij}}\left|x^*{}_j\right| + (\Gamma - \lfloor\Gamma\rfloor)\widehat{a_{it}}|x^*{}_t|\right\} \leq b_i, \tag{3}$$

where it is equal to the objective function of the following linear optimization problem:

$$\begin{aligned} &\beta(x^*,\Gamma) = \max \sum_{j\in J} \widehat{a_{ij}}\left|x^*{}_j\right| R_{ij}, \\ &\sum_{j\in J} R_{ij} \leq \Gamma, \\ &0 \leq R_{ij} \leq 1 \qquad \forall j \in J. \end{aligned} \tag{4}$$

The dual model of the above problem is:

$$\begin{aligned} &\min \sum_{j\in J} \rho_{ij} + \Gamma Z_i, \\ &\rho_{ij} + Z_i \geq \widehat{a_{ij}}\left|x^*{}_j\right| && \forall i.j \in J, \\ &\rho_{ij} \geq 0 && \forall j \in J, \\ &Z_i \geq 0 && \forall i. \end{aligned} \tag{5}$$

By strong duality, since *Problem (4)* is feasible and bounded for all $\Gamma \in [0, |J|]$, then the dual *Problem (5)* is also feasible and bounded and their objective values coincide. So we have that $\beta(x^*, \Gamma)$ is equal to the objective function value of *Problem (5)*.

Now *Model (2)* can be reformulated as a linear optimization model as follows [37]:

$$\begin{aligned} &min\, t, \\ &\sum_{j\in J} \overline{a_{0j}}x_j + \Gamma.Z_0 + \sum_{j\in J} \rho_{0j} \leq t, \\ &\sum_{j\in J} \overline{a_{ij}}x_j + \Gamma.Z_i + \sum_{j\in J} \rho_{ij} \leq b_i && \forall i \in I, \\ &Z_i + \rho_{ij} \geq \widehat{a_{ij}}x_j && \forall j \in J.i \in I \cup \{0\}, \\ &Z_i.\rho_{ij} \geq 0 && \forall j \in J.i \in I \cup \{0\}, \end{aligned} \tag{6}$$

where Γ shows the degree of conservatism.

3. Problem Statement

Consider a set of n jobs $(i=1, 2,...,n)$ that have to be processed at S stages in sequence. Each job i consists of a chain of operations. An operation should be processed at stage j $(j=1, 2, ..., S)$ for which there are m_j $(m_j \geq 1)$ identical parallel machines. At least one stage has at least two parallel machines. All jobs have the same machine sequence to be processed and there is a set of m $(K=1, 2, ..., m)$ total machines at all stages. Since all parallel machines within each stages are identical, the processing time of an assigned job to a machine does not depend on the specific machine (*Fig. 1*).

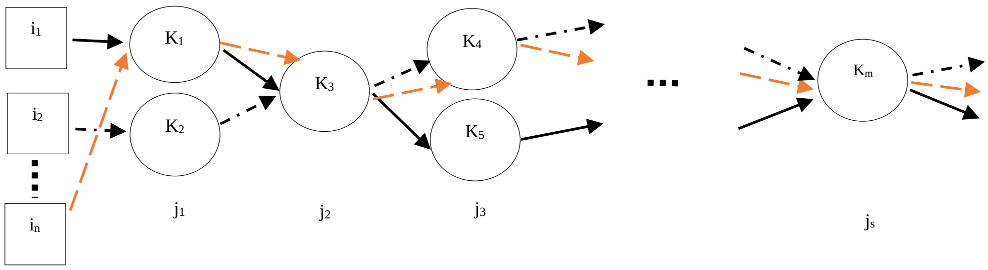

Fig. 1. *Hybrid flow shop scheduling problem.*

Wagner's position-based modelling method [16] is applied in the present paper to develop an HFS model. Hence, we divided each machine into several processing positions according to the time sequence. One position cannot be assigned to more than one job simultaneously. A scheduling plan should arrange the jobs to processing positions of the machines.

To develop the model, *Table 2* and *Table 3* show the assumptions and a description of the notations.

Table 2. *Assumptions.*

Number	Description
1	*At least one stage has at least two parallel machines.*
2	*All the machines have two modes (on and off mode).*
3	*Machines are automatically operated and labor is not included.*
4	*All machines are available at time zero.*
5	*Parallel machines at each stage are unrelated.*
6	*Parallel machines at each stage in terms of capacity, electricity consumption and processing speed, are identical.*
7	*Parallel machines at one stage are independent of other stages.*
8	*Each machine can only process one job at any time and when it is processing a job.*
9	*Machines are always available.*
10	*Breakdown and preventive maintenance are not considered.*
11	*Preparation and movement time between machines are not considered.*
12	*There are j different jobs.*
13	*All jobs are available at time zero and jobs release times are zero.*
14	*Job processing cannot be interrupted, until the completion of job. No interruption once a job has started (non-preemption).*
15	*A job cannot be processed simultaneously on more than one machine.*

Table 3. *A description of notations used in all formulas.*

Notation	Description
i, ii	*Jobs index.*
j, jj	*Stages index.*
k	*Machines index.*
p	*Positions index.*
n	*Number of jobs.*
s	*Number of stages.*
m	*Total number of machines.*
m_j	*Number of parallel machines within stage j, mj ≥ 1.*
I	*Set of jobs where {1, 2,..., n}.*
J	*Set of stages where {1, 2,..., S}.*
K_j	*Set of parallel machines within stage j where {1,2,...,mj}.*
K	*Set of total machines where {1, 2,...,m}.*
P	*Set of positions of machine k where {1,2,...,n}.*
M	*A very large positive integer.*
p_{ik}	*processing time of job i on machine k.*
d_i	*Due date of job i.*
Z_{jk}	*Parameter that is equal 1 if machine k is at stage j, and 0 otherwise.*
B_{ij}	*Continuous variable that determines the starting time of jth operation of job i.*
S_{kt}	*Continuous variable that determines the starting time of machine k in position t.*
T_i	*Tardiness of job i.*
Cmax	*The makespan.*
x_{ik}	*Binary variable that is equal 1 if job i is processed on machine k, and 0 otherwise.*
y_{ikp}	*Binary variable that is equal 1 if job i occupies position p of machine k, and 0 otherwise.*

3.1. MILP Model

We consider the following extended model of Meng et al. [20] that is bi-objective and takes into account uncertainty in processing times and due date:

$$\text{Min } C_{\max,} \tag{7}$$

$$\text{Min } \sum_i T_{i,} \tag{8}$$

S.t.

$$\sum_{k \in K_j} x_{i.k} = 1 \qquad \forall i.j, \tag{9}$$

$$\sum_{p \in P} y_{i.k.p} = x_{i.k} \qquad \forall i.k \in K, \tag{10}$$

$$\sum_i y_{i.k.p} \leq 1 \qquad \forall k \in K.p \in P, \tag{11}$$

$$\sum_i y_{i.k.p} \geq \sum_{ii} y_{ii.k.p+1} \qquad \forall k \in K.p \in \{1.2.\dots.n-1\}, \tag{12}$$

$$B_{i.j} + \sum_{k \in K_j} P_{i.k} x_{i.k} \leq B_{i.j+1} \qquad \forall i.1 \leq j \leq s-1, \tag{13}$$

$$S_{k.p} + \sum_i P_{i.k} y_{i.k.p} \leq S_{k.p+1} \qquad \forall k \in K.1 \leq p \leq n-1, \tag{14}$$

$$S_{k.p} \leq B_{i.j} + M(1 - y_{i.k.p}) \qquad \forall i.j.k \in K_j.p \in P, \tag{15}$$

$$S_{k.p} \geq B_{i.j} - M(1 - y_{i.k.p}) \qquad \forall i.j.k \in K_j.p \in P, \tag{16}$$

Where, the *Objectives (7)* and *(8)* are to minimize the makespan and total tardiness,

$$C_{max} \geq B_{i.S} + \sum_{k \in K_S} P_{i.k} x_{i.k} \quad \forall i, \quad (17)$$

$$B_{i.S} + \sum_{k \in K_S} P_{i.k} x_{i.k} - d_i \leq T_i \quad \forall i, \quad (18)$$

$$T_i . B_{i.j} . S_{k.p} \geq 0, \quad (19)$$

$$x_{i.k} . y_{i.k.p} \in \{0.1\}. \quad (20)$$

respectively. *Constraint (9)* determine that each job cannot be assigned to more than one machine at each stage. The relation between the two decisions variables are defined by *Constraint (10)*. *Constraint (11)* represent that any position of a machine processes exactly one job at a time. *Constraint (12)* guarantee that jobs assignment to the machine positions are in sequential order. That is, each position of machines can only be occupied when its previous position of the same machine is occupied. *Constraint (13)* are associated with the operation precedence relations. They guarantee that each operation of each job can be started only if its precedent operation has been completely finished at the previous stage. Based on the *Constraint (14)*, each machine has to process at most one operation at a time. That is, for two adjacent operations assigned to the same machine, the succeeding operation can be only started when the precedent operation has been completely finished. *Constraints (15)* and *(16)* guarantee that the starting time for a position of a machine equals to the starting time of the operation arranged at this position. *Constraint (17)* calculate the makespan and *Constraint (18)* calculate the tardiness of each job. *Constraints (19)–(20)* also define the decision variables.

3.2. Robust Formulation

Suppose that *Model (2)* has two uncertain parameters $\widetilde{p_{\imath k}} \in [\overline{P_{\imath k}} - \widehat{P_{\imath k}} . \overline{P_{\imath k}} + \widehat{P_{\imath k}}]$ and $\widetilde{d_\imath} \in [\overline{d_\imath} - \widehat{d_\imath} . \overline{d_\imath} + \widehat{d_\imath}]$ where $\overline{P_{\imath k}}$ and $\overline{d_\imath}$ are the nominal and $\widehat{P_{\imath k}}$ and $\widehat{d_\imath}$ are the perturbation value of these parameters. These parameters cause uncertainty in the *Constraints (13), (14), (17)* and *(18)*. *Constraints (13)* and *(17)* contain $\sum_{k \in K_j} \tilde{P}_{i.k} x_{i.k}$. We consider parameter $\Gamma \in [0.m_j]$ to adjust robustness of the model against the level of conservatism of the solution. The uncertain coefficients set is also K_j ($m_j = |K_j|$). Furthermore, *Constraints (18)* calculate the tardiness of each job *i* and contains $\tilde{d}_i$. Here also, we set parameter $\Gamma'' \in [0, 1]$. Hence, the followings are the robust counterparts of the *Constraints (13), (17)* and *(18)* (based on *Model (6)*):

$$B_{ij} + \sum_{k \in K_j} \overline{P_{\imath k}} x_{ik} + \sum_{k \in K_j} \rho_{ijk} + \Gamma Z_{ij} \leq B_{ij+1} \quad \forall\, i.\, 1 \leq j \leq s-1, \quad (21)$$

$$\rho_{ijk} + Z_{ij} \geq \widehat{P_{\imath k}} x_{ik} \quad \forall\, i.\, 1 \leq j \leq s.\, k \in K_{j,} \quad (22)$$

$$\rho_{ijk} . Z_{ij} \geq 0 \quad \forall\, i.\, 1 \leq j \leq s\, .\, k \in K_j, \quad (23$$

$$B_{is} + \sum_{k \in K_s} \overline{P_{\imath k}} x_{ik} + \sum_{k \in K_s} \rho_{isk} + \Gamma Z_{is} \leq Cmax \quad \forall i, \quad (24)$$

$$B_{is} + \sum_{k \in K_s} \overline{P_{\imath k}} x_{ik} + \sum_{k \in K_s} \rho_{isk} + \Gamma Z_{is} - \left(\overline{d_\imath} - \Gamma'' \widehat{d_\imath}\right) \leq T_i \quad \forall\, i. \quad (25)$$

Constraint (14) contain $\sum_i P_{i.k} y_{i.k.p}$ and parameter $\Gamma' \in [0.n]$. The uncertain coefficients set is I ($n = |I|$). In fact, by assigning the value to $\Gamma . \Gamma'$ and Γ'' in all the robust counterparts, a trade-off occurs between the robustness and optimality of the problem. These parameters are determined by the decision maker and their values are dependent on the risk aversion and the importance of the constraint for the decision maker [32, 33]. Based on *Model (6)* that demonstrates the robust counterpart of a constraint (resulted of B&S approach), *Constraints (26)-(28)* enter the problem model instead of the *Constraint (14)* as follows:

$$S_{kp} + \sum_i \overline{P_{ik}} y_{ikp} + \sum_i \rho'_{ikp} + \Gamma' Z'_{kp} \le S_{kp+1} \qquad \forall\, k.\, 1 \le p \le n-1, \tag{26}$$

$$\rho'_{ijk} + Z'_{kp} \ge \widehat{P_{ik}} y_{ikp} \qquad \forall\, i.\, j.\, 1 \le p \le n-1, \tag{27}$$

$$\rho'_{ijk}.\, Z'_{kp} \ge 0 \qquad \forall\, i.\, j.\, 1 \le p \le n-1. \tag{28}$$

If the value of parameters Γ, Γ' and Γ'' equal to 0, the robust problem reduces to the nominal problem.

4. Multi Objective Optimization by Fuzzy Goal Programming

One of the methods that is widely used in the multi-objective problems is goal programming that presents the efficient solution set. However, selection of the satisfactory solution among this set is not easy for the decision makers. Therefore, the Fuzzy Goal Programming (FGP) introduced by Narasimhan [38] applied by many researchers to solve multi-objective problems [39]-[43].

Consider the following multi-objective model:

$$\begin{gathered} \text{Min}\ \left(f_1(x).\, f_2(x).\, \ldots .\, f_n(x) \right) \\ x \in X. \end{gathered} \tag{29}$$

Based on the idea of FGP, each objective $f_i(x)$ has an associated fuzzy goal which deviations of all goals should be minimized. The membership functions of each fuzzy goal represent the grade of membership of a goal in the fuzzy subset and FGP approach aims to minimize the deviational variables to achieve the highest degree of each membership goals and also the most satisfactory solution [39].

The membership function of each goal is defined based on the Positive-Ideal Solution (PIS) and Negative-Ideal Solution (NIS), which respectively are the best possible value and the feasible worst solution of each objective function. Therefore, the satisfactory degree of 1 is assigned to the PIS as the most preferred value and NIS has the satisfactory degree of 0. Consider m_i as the minimum value and M_i as the maximum value of objective $f_i(x)$. Since all objective functions of the developed MILP model in the present study are of minimization-type, m_i is the PIS for each $f_i(x)$ and has the satisfactory degree of 1 and M_i is the NIS for each $f_i(x)$ and has the satisfactory degree of 0.

It should be noted that if we define the membership functions of the deviations of the goals instead of the goals, the minimax FGP can be applied. Thereby, the objective function of FGP is minimization of the maximum of deviations. Based on this method, the difference between PIS (m_i) and NIS (M_i) calculates the acceptable deviation from the goal. Then, the membership function can be shown as *Fig. 2* and presented by

$$\mu(x)_{|f_i} = \begin{cases} 0 & f_i(x) \geq M_i \\ \dfrac{f_i(x)-m_i}{M_i - m_i} & x \in X \quad . \\ 1 & f_i(x) \leq m_i \end{cases} \tag{30}$$

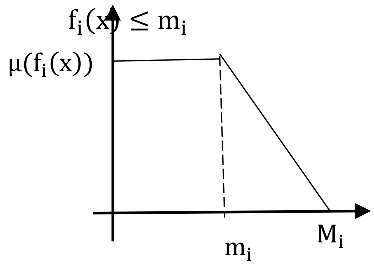

Fig. 2. *Membership function of deviations.*

Therefore, the multi-objective *Model (29)* can be defined as a single objective problem as follows:

$$\min z = \max\{d_i(x){:}\, i = 1{:}\, n\},$$

$$d_i(x) = \frac{f_i(x) - m_i}{M_i - m_i} \; ; \; i = 1.2. \dots . n. \tag{31}$$
$$x \in X$$

It can be clearly seen that if $f_i(x) \to m_i$, then $d_i(x) \to 0$. The above model can be replaced by *Model (32)* which is linear:

$$min\ Z,$$

$$Z \geq \frac{f_i(x) - m_i}{M_i - m_i} \; ; \; i = 1.2. \dots . n. \tag{32}$$
$$x \in X$$

5. Computational Results

To evaluate the efficiency of the developed model, some instances are generated. Initially, we solved the problem with the nominal values. Then, $\Gamma \in [0, m_j]$, $\Gamma' \in [0, n]$, and $\Gamma'' \in [0, 1]$ are determined based on B&S approach. Finally, the perturbation value of uncertain parameter, i.e., the processing time and due date are defined as $\widehat{P_{ik}} = \alpha \overline{P_{ik}}$ and $\widehat{d_i} = \beta \overline{d_i}$ (α and β are predetermined).

5.1. Design of Experiments

The data required to create the problems are shown in *Table 4*. The characteristic of each problem is introduced by A-B-C-D, which shows, respectively, the number of jobs, the number of stages, the number of total machines and the number of machine positions.

Table 4. *Problems with nominal parameters values given to GAMS.*

Number of Instance	Problem Characteristic	Number of Machines per Stages	Nominal Processing Times	Nominal Due Dates
1	*5-2-3-5*	*2* *1*	*[5-10]* *[4-6]*	*U[10-30]*
2	*7-2-3-7*	*2* *1*	*[5-10]* *[4-6]*	*U[20-50]*
3	*8-3-5-8*	*2* *1* *2*	*[20-70]* *[20-60]* *[10-40]*	*U[100-500]*

To solve both the deterministic and robust models of problems given in *Table 3*, CPLEX solver of GAMS software is used in a Laptop with Intel Core i7- 2.6 GHz processor and 8 GB RAM. Moreover, the maximum runtime of the solver is set to 2000 seconds. *Table 5* shows the objectives and their deviations that have been obtained after solving the final developed model by applying FGP approach. The results are presented for the perturbation $\alpha = 0.2$ and $\beta = 0.1$(Robust1) and $\alpha = 0.3$ and $\beta = 0.2$ (Robust2). Moreover, we set Γ=1.5, $\Gamma' = 3$ and $\Gamma'' = 0.7$.

***Table 5.** Computational results.*

Instance	Objectives	Deterministic	Robust 1 ($\alpha = 0.2$ $\beta = 0.1$)	Robust 2 ($\alpha = 0.3$ $\beta = 0.2$)
	Cmax	*25*	*27.1*	*29.7*
	Tardiness	*11.277*	*28.186*	*33.37*
1	*Devation of Cmax*	*0.09*	*0.108*	*0.201*
	Devation of Tardiness	*0.09*	*0.108*	*0.201*
	Cmax	*39*	*42.127*	*48.605*
	Tardiness	*11.2*	*18.683*	*47.610*
2	*Devation of Cmax*	*0.154*	*0.162*	*0.194*
	Devation of Tardiness	*0.132*	*0.111*	*0.194*
	Cmax	*347.036*	*409.226*	*442*
	Tardiness	*161.680*	*255.973*	*434.477*
3	*Devation of Cmax*	*0.469*	*0.102*	*0.2*
	Devation of Tardiness	*0.469*	*0.102*	*0.05*

Figs. 3 (a)-3(c) also plot the objective values of the problems with these three levels of perturbations. It can be found that increasing the perturbation of uncertain parameters not only lead to increasing the objectives, but also has a stronger effect on Tardiness compared to C_max. In fact, these figures illustrate the sensitivity analysis of objectives to level of perturbation of the uncertain parameters.

To compare the impact of uncertainty on scheduling plan, *Figs. 4* and *5* show the Gantt charts of the obtained solutions for *Problem 2* (7-2-3-7) with and without uncertainty of parameters. As can be seen, uncertainty not only leads to greater maximum of completion times (it is 42.5 when we do not consider the uncertainty and 54.6 when the uncertainty is considered), but also changes the assignment of jobs to machines and their sequential order. For example, at stage 1, job 5 is assigned to machine k_1 without and to machine k_2 with uncertainty. Moreover, this job is assigned to 2nd machine position without and 3rd machine position with uncertainty. It should be noted that the developed robust counterpart formulation in this paper is capable to guarantee the feasibility. Hence, a schedule plan is created so that the uncertainty of data leads to the least possible modifications.

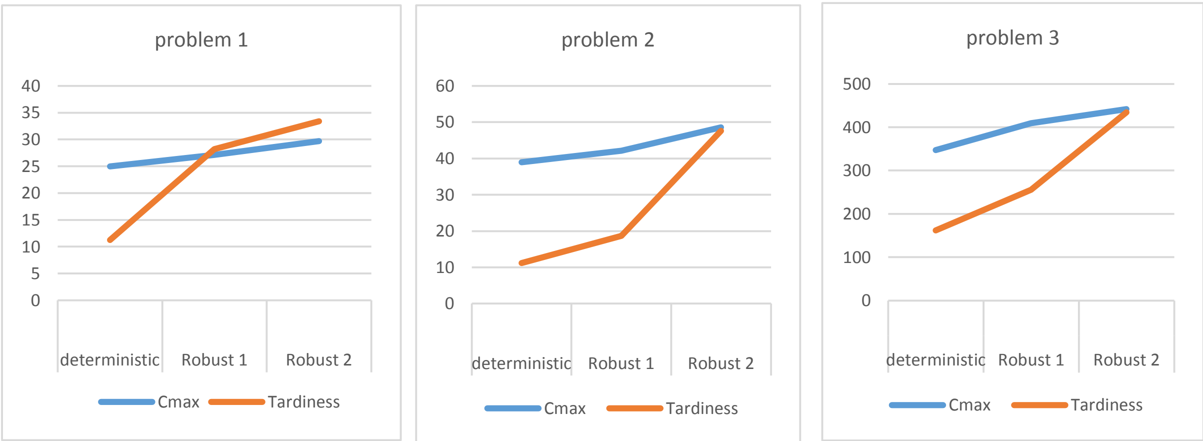

Fig. 3. *The effect of perturbation of uncertain parameters on objective functions.*

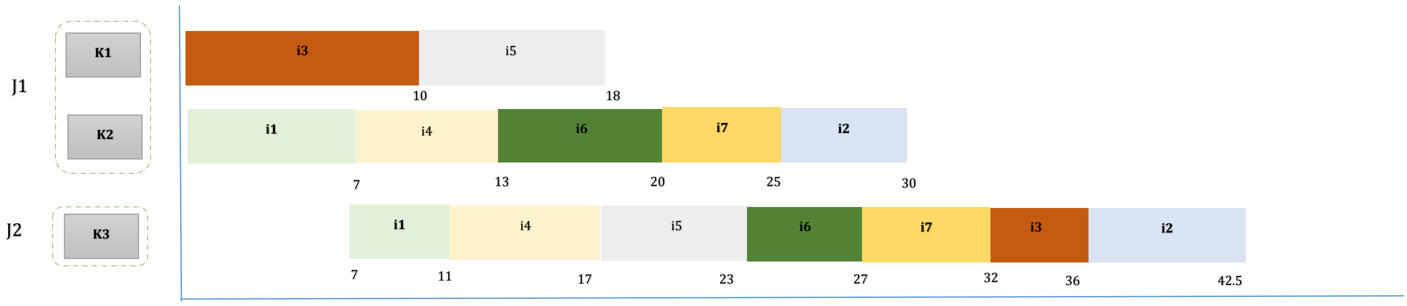

Fig. 4. *The robust schedule of problem 2 without uncertainty of parameters.*

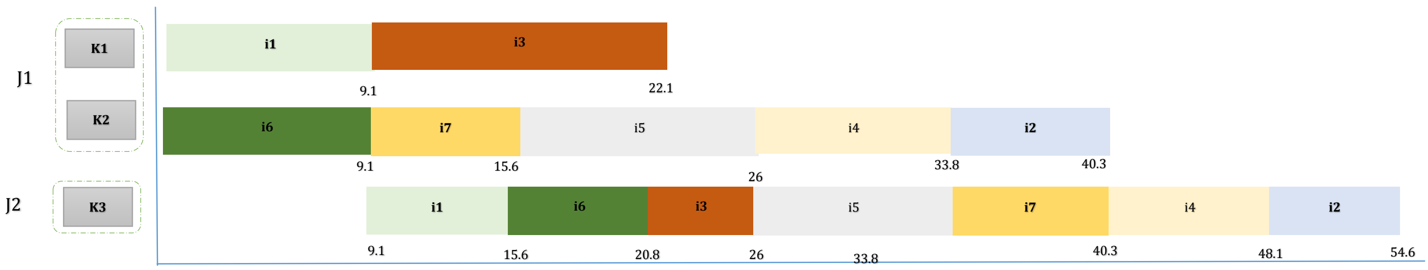

Fig. 5. *The robust schedule of problem 2 with uncertainty of parameters.*

Besides the above experiments, the robust counterpart problem is solved for the different degrees of conservatism. The effect of changing the protection level on the objective function values for the problem 2 with $\Gamma \in [0.2]$, $\Gamma' \in [0.7]$ *and* $\Gamma'' \in [0.1]$ is illustrated in *Fig. 6*. As it can be found, tardiness objective function is more affected by changing the conservatism level. Hence, according to *Fig. 6 (b)* compared to the other charts, when protection level increases, the objective values increase more. That is, the more jobs (n) process with uncertain time, the larger the completion time (C_max) and the larger delay in due date (Tardiness). Moreover, Γ'' is associated with due date based on which the tardiness objective function is claculated. So, different levels of this parameter strongly affects the tardiness compared to the C_max as shown in *Fig. 6 (c)*.

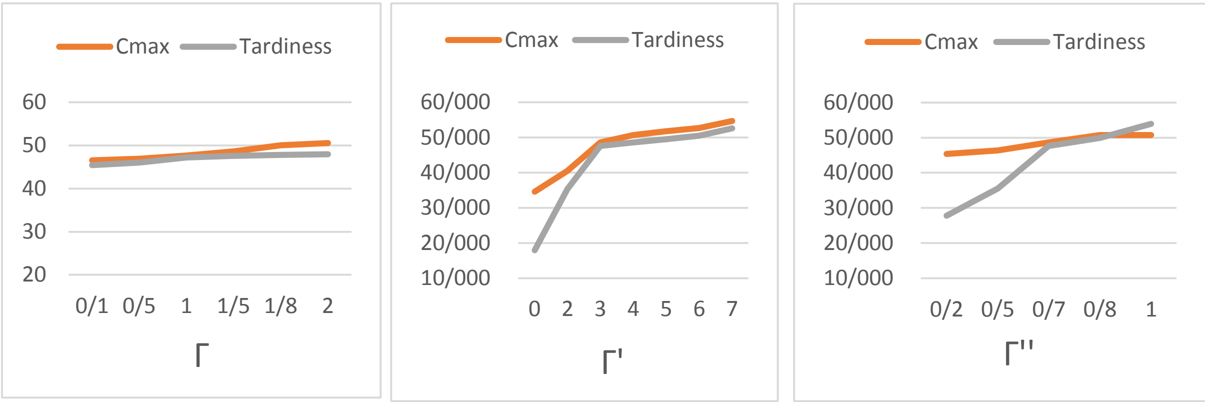

Fig. 6. *Objective values as a function of conservatism levels for Problem 2.*

For further sensitivity analysis and to have better understanding of the objective functions behavior, we compare the effect of robust optimization on the objective values of *Problem 2* (7-2-3-7). The deterministic and robust models with 20% perturbation for all uncertain parameters ($\alpha = \beta = 0.2$) are solved. *Table 6* shows the obtained objective values and deviations for three states: without any protection ($\Gamma = \Gamma' = \Gamma'' = 0$), with the maximum protection ($\Gamma = 2$, $\Gamma' = 7$ and $\Gamma'' = 1$) and with the predetermined protection ($\Gamma = 1.5$, $\Gamma' = 3$ and $\Gamma'' = 0.7$). In the absence of protection of the processing time and due date, the values of C_max and Tardiness are 39 and 11.2. However, with maximum protection that turns to Soyster's [35] method, the objective values is increased. Compared to the maximum protection, when we reduce the conservatism levels to a predetermined protection mentioned above, the objective values change and become better. In fact, reducing the conservatism level leads to improve the objective functions. As mentioned before, the FGP approach that is applied to solve this two-objective problem tends to minimize the deviations of ideal solution. Furthermore, we have three parameters as conservatism level with different range values. Therefore, it is reasonable that reducing all the conservatism levels simultaneously may affect the objective values in various behaviors. However, we can be sure that this final solution has the minimum deviation from the ideal solution for all objectives.

Based on the above discussion, it is clear that changing the protection level effects the objective values. An important observation is that B&S formulation succeeds in reducing the price of robustness, that is, we do not heavily penalize the objective function value in order to protect ourselves against constraint violation.

Table 6. *Objective values for different degrees of conservatism.*

Problem	Objectives	Deterministic	Predetermined Protection	Maximum Protection
7-2-3-7	*Cmax*	*39*	*39.6*	*43.2*
	Tardiness	*11.2*	*16.7*	*27.335*

6. Conclusion

One of the complicated scheduling problems that has good adaptability with real production systems is the HFS environment. This paper suggested a multi-stage HFS model under uncertain processing time and due date with objectives aim to minimize the total tardiness and makespan. The robust optimization based on Bertsiams and Sim approach was applied in order to deal with the uncertainty and Fuzzy Goal Programming is also implemented to solve this multi-objective problem. Based on the computational results of solving three different-sized instance problems, this model was capable to find the satisfactory sequence of jobs for each machine at each stage so that the uncertainty of data leads to the least possible modifications. However, uncertainty changed the assignment of jobs to machines and their sequential order. Moreover, the larger the level of perturbation was, the larger objective function values of the solutions were. Increasing the degree of conservatism also made the objectives to become worse and effected the tardiness stronger than makespan. Besides, the total tardiness of jobs is more affected by changing the protection level of the due date than the processing time. It should be noted that the CPLEX solver of GAMS could solve the extended RMILP model of HFS for the small and medium size problems. It could obtain the satisfactory solution within the time limit and raise the lower bounds in order to prove the optimality of the solution.

One may consider the followings as the future research directions:

- Adding further conditions to the RMILP model such as sequence dependent setup time, precedence constraints between operations from different jobs, preemptions, machine break down, no-wait jobs, new job arrivals.
- Extending the developed model to uniform and unrelated parallel machines.

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Evaluation of Domestic Energy Preference in Nigeria Cities: a Case Study of Warri, Benin, Port Harcourt, and Calabar

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ABSTRACT

The significance of energy in human lives cannot be overemphasized. It is crucial to all facets of economic growth, progress, and development as well as poverty eradication and security. Household energy is generally required for a variety of purposes. It is required for lighting and heating. This study evaluated domestic energy utilization in major Nigeria cities using Warri, Benin, Port Harcourt, and Calabar as a case study. A total of four locations with the highest population were targeted at each selected city. Responses were randomly selected from energy utilization in the selected region. A total of 1439 questionnaires were administered all through the selected cities in this study. Data obtained were evaluated using descriptive statistic. The result obtained revealed that kerosene has the highest percentage of utilization (88%), LNG (68.9%), electricity (67.7%), PMS (67.1), diesel (5.6%), charcoal (16.4%), wood (28.2%), and solar (3%) within these cities. Therefore, kerosene is the most preferred energy and this can be as a result of cost when compared to LNG, electricity, PMS, diesel, charcoal, solar, and wood fuel.

1. Introduction

Energy can be classified into solid fuels and non-solids fuel. The solid fuel includes fossil fuel (coal, peat) and biomass (wood, dungs and agricultural product) while the non-solid fuel consists of kerosene, liquefied natural gas (LNG), etc. [1]. Household energy is the energy utilized for domestic purposes. Domestic energy usage is generally influence by various factors that include standard of living of the country, climate, residential type, occupants' age, and education level [2]-[4]. Increase demand for energy consumption has adverse effect on the income of individual and this has resulted to high amount of money spent on energy utilized [5]-[7]. More so, high demand for biomass fuel have led to an increase in deforestation at a rate of about 400.000 hectares per year [8]. If this trend continues, the country forest resources could be completely depleted [9]. Besides, the looming exhaustion of energy sources,

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incessant power supplies, and the quest to meet the growing sustainable energy demand for the global population had been a huge motivation factor for research attention [14] and [15].

Energy is a key factor to human activities and it is quite critical to social and economic development. Energy is one of the basic input in production process and is a key factor in developing nations [16]. The importance of energy in human lives cannot be overemphasized; it is central to all aspects of economic growth, progress and development as well as poverty eradication and security [12]. It is a vital commodity that is requires for the existence of modern household living [13] and [14]. It is recognized that energy play a significant role in the economic development of a country as it improves the productivity of the nation. As a result of the nature of most developing nations and because of inadequacy of understanding of domestic dynamic and other factors influencing it dynamism, it is difficult to fully identify the impact of insufficient energy utilize on the development of the various region in the country and the standard of living of the people.

Furthermore, there is a problem in designing policies intended to address the impact of the use of energy within this sector. Also, reliability and affordability of energy is a major challenge in this region. In Nigeria, energy use in the domestic sector account for more than 60% to final energy utilize [10]. Most of these energy is utilized for domestic activities such as cooking, lighting and home electrical appliances. A century ago, the problem of household energy demand was not common as Nigerians were mostly engage in farming and the population growth was very small. Technological advancement was not well visible and the modern equipment which use exorbitant energy were not discovered in Nigeria. Utilization of energy is increasing drastically in Nigeria cities with respect to increasing human population and higher living standard with negative impacts on the environment. This study therefore tends to evaluate the current trends in domestic energy utilization in selected cities within the Southern part of Nigeria.

2. Research Methodology

The data used for this study was collected via direct interview and structured questionnaires. The data collected is based on the response from household energy utilized. The study was carried out in Warri, Benin, Port Harcourt and Calabar. These cities are located as Warri (5.5544N,5.793E), Benin (6.3350N,5.6037E), Port Harcourt (4.8156N,7.0498E) and Calabar (4.9757N,8.3417E) with a corresponding population of 814,000,1,727,000,1,865,000 and 579,000 respectively [11]. The reason for the selection of these cities were due to their high demand of energy. Kerosene which is commonly use among household have been a scare energy source. The scarcity is as a result of the subsidy removed by the Nigeria Federal Government (NFG). This on the other hand has resulted in high numbers of illegal mining of crude oil, a process commonly referred to as “Oil Bunkering” in the region. These illegal activities have led to pollution of the environment and loss of human lives.

3. Results and Discussion

The results of the summary of survey per selected cities are shown in *Table 1*.

Table 1. *Summary of survey per city in southern Nigeria.*

S/N	City	No. of Questionnaires Propose	No. of Questionnaires Utilized	Percentage
1	*Warri*	*350*	*334*	*95.4*
2	*Benin*	*400*	*386*	*96.5*
3	*Port Harcourt*	*450*	*432*	*96.4*
4	*Calabar*	*300*	*285*	*95.0*

Table 2 shows the outcome of the analysis of respondents on household energy utilization in Warri.

Table 2. *Analysis of respondents on household energy utilization in Warri.*

S/N	Location	No. of Respondent	Electricity	LNG	Wood Fuel	PMS	Kerosene	Diesel	Charcoal	Solar
1	*Ugborikoko*	*84*	*45*	*53*	*40*	*35*	*75*	*0*	*42*	*0*
2	*DDPA*	*42*	*40*	*42*	*10*	*42*	*40*	*5*	*0*	*4*
3	*NPA Quarter*	*54*	*43*	*54*	*5*	*50*	*53*	*2*	*3*	*3*
4	*Niger Cat/Chickelly*	*154*	*113*	*120*	*45*	*87*	*130*	*4*	*23*	*2*
	Total	*334*	*241*	*269*	*100*	*214*	*298*	*11*	*68*	*9*
	Percentage	*100*	*72.2*	*80.5*	*29.9*	*64.1*	*89.2*	*3.3*	*20.4*	*2.7*

Table 3 shows the outcome of the analysis of respondentson household energy utilization in Benin City.

Table 3. *Analysis of respondents on household energy utilization in Benin city.*

S/N	Location	No. of Respondent	Electricity	LNG	Wood Fuel	PMS	Kerosene	Diesel	Charcoal	Solar
1	*Ugbowo*	*125*	*86*	*72*	*45*	*68*	*118*	*10*	*52*	*3*
2	*Sapele Road*	*86*	*72*	*53*	*36*	*76*	*82*	*5*	*33*	*0*
3	*Akpakpava*	*95*	*63*	*41*	*25*	*53*	*88*	*7*	*42*	*5*
4	*GRA*	*80*	*75*	*80*	*5*	*80*	*65*	*15*	*5*	*6*
	Total	*386*	*296*	*246*	*111*	*277*	*353*	*37*	*132*	*14*
	Percentage	*100*	*76.7*	*63.7*	*28.8*	*71.8*	*91.5*	*9.6*	*34.2*	*3.6*

Table 4 shows the results of the analysis of respondents on household energy utilization in Port Harcourt.

Table 4. *Analysis of respondents on household energy utilization in Port Harcourt.*

S/N	Location	No. of Respondent	Electricity	LNG	Wood Fuel	PMS	Kerosene	Diesel	Charcoal	Solar
1	*Diobu*	*125*	*68*	*88*	*35*	*86*	*112*	*6*	*10*	*5*
2	*Rukpokwu*	*127*	*72*	*76*	*28*	*83*	*98*	*3*	*7*	*4*
3	*Woji Estate*	*86*	*46*	*37*	*15*	*67*	*76*	*2*	*9*	*0*
4	*GRA Phase 1-5*	*96*	*43*	*72*	*18*	*53*	*83*	*5*	*8*	*3*
	Total	*434*	*229*	*273*	*96*	*289*	*369*	*16*	*34*	*12*
	Percentage	*100*	*52.8*	*62.9*	*22.1*	*66.6*	*85*	*3.7*	*7.8*	*3.0*

Table 5 shows the results of the analysis of respondents on household energy utilization in Calabar.

Table 5. *Analysis of respondents on household energy utilization in Calabar.*

S/N	Location	No. of Respondent	Electricity	LNG	Wood Fuel	PMS	Kerosene	Diesel	Charcoal	Solar
1	*State Housing*	*56*	*43*	*42*	*15*	*48*	*53*	*4*	*6*	*3*
2	*Federal House*	*67*	*51*	*46*	*21*	*38*	*62*	*2*	*10*	*0*
3	*Satellite Town*	*82*	*58*	*69*	*28*	*57*	*65*	*3*	*12*	*2*
4	*Etta Agbo*	*80*	*56*	*46*	*35*	*43*	*71*	*2*	*8*	*2*
	Total	*285*	*208*	*203*	*99*	*186*	*251*	*11*	*36*	*7*
	Percentage	*100*	*73*	*71.2*	*34.7*	*65.3*	*88.1*	*3.9*	*12.6*	*2.5*

Table 6 shows the summary of the analysis of respondents on household energy utilization in the Nigeria cities.

Table 6. *Summary of analysis of respondents on household energy utilization in the selected cities.*

S/N	Location	No. of Respondent	Electricity	LNG	Wood Fuel	PMS	Kerosene	Diesel	Charcoal	Solar
1	*Warri*	*334*	*241*	*269*	*100*	*214*	*298*	*11*	*68*	*9*
2	*Benin City*	*386*	*296*	*246*	*111*	*277*	*353*	*37*	*132*	*14*
3	*Port Harcourt*	*434*	*229*	*273*	*96*	*289*	*369*	*16*	*34*	*12*
4	*Calabar*	*285*	*208*	*203*	*99*	*186*	*257*	*17*	*36*	*7*
	Total	*1439*	*974*	*991*	*406*	*966*	*1277*	*81*	*236*	*42*
	Percentage	*100*	*67.7*	*68.9*	*28.2*	*67.1*	*88.7*	*5.6*	*16.4*	*3*

Table 7 shows the income level of respondents in relation to their choice of household energy utilization in the four southern cities.

Table 7. *Respondent income level in relation to their choice of household energy utilizeation in the selected cities.*

S/N	Monthly Income (₦)	No. of Respondent	Electricity	LNG	Wood Fuel	PMS	Kerosene	Diesel	Charcoal	Solar
1	*Below 50,000*	*652*	*334*	*204*	*285*	*283*	*635*	*0*	*1486*	*0*
2	*Above 50,000*	*465*	*382*	*465*	*232*	*386*	*386*	*0*	*63*	*0*
3	*100,000-300,000*	*256*	*193*	*256*	*111*	*234*	*237*	*23*	*17*	*8*
4	*Above 300,000*	*66*	*65*	*66*	*5*	*63*	*119*	*58*	*8*	*35*
	Total	*1439*	*974*	*991*	*406*	*966*	*1277*	*81*	*236*	*42*
	Percentage	*100*	*67.7*	*68.9*	*28.2*	*67.1*	*88.7*	*5.6*	*16.4*	*3*

As shown in *Fig. 1* is the graphical analysis of respondent income level in relation to their energy preference while *Fig. 2* shows the respondent analysis on household energy utilization in selected cities of Nigeria. From the analysis, kerosene with 88.7% (*Table 7*) is widely used in the selected cities used for this study. There was a continuous reliance on the use of kerosene and this was due to the relatively low cost of the product when compared to other products used in this study. The price of kerosene varies around at a very low price of ₦120 per litre which is relatively cheaper than LNG and electricity as an energy for household usage. The removal of subsidy by the government from kerosene have led to an increase in illegal mining (Oil bunkering) of the product in region. Besides, 68.9% respondents indicated their preference for LNG as their primary energy usage. Thus, there is a high demand of LNG as a domestic energy source among cities dweller in this present day as compared to decade ago. This further affirm the fact that users over the years have been enlighten on the superior nature of this energy source. Also, the use of LNG as household energy is faster and more convenient when compared with other cooking energy sources.

Furthermore, for electricity (67.7%), PMS (67.1%) and diesel (5.6%), findings from the study suggest electricity source is in high demand. Athough, Nigeria power generation and Distribution has been a major problem in the energy sector over the years which the Government has failed to improve on. Most areas in these cities have an average of four (4) hours supply of electricity daily, and this has led to high demand of PMS and diesel in order to meet up with daily energy needs. Diesel has the lowest percentage from the respondents' analysis, and this is attributed to the high cost of running diesel generators compared to PMS power generators. More so, there is a high demand of biomass fuel (charcoal=16.4%), wood = 28.2%) as shown in *Table 7*.

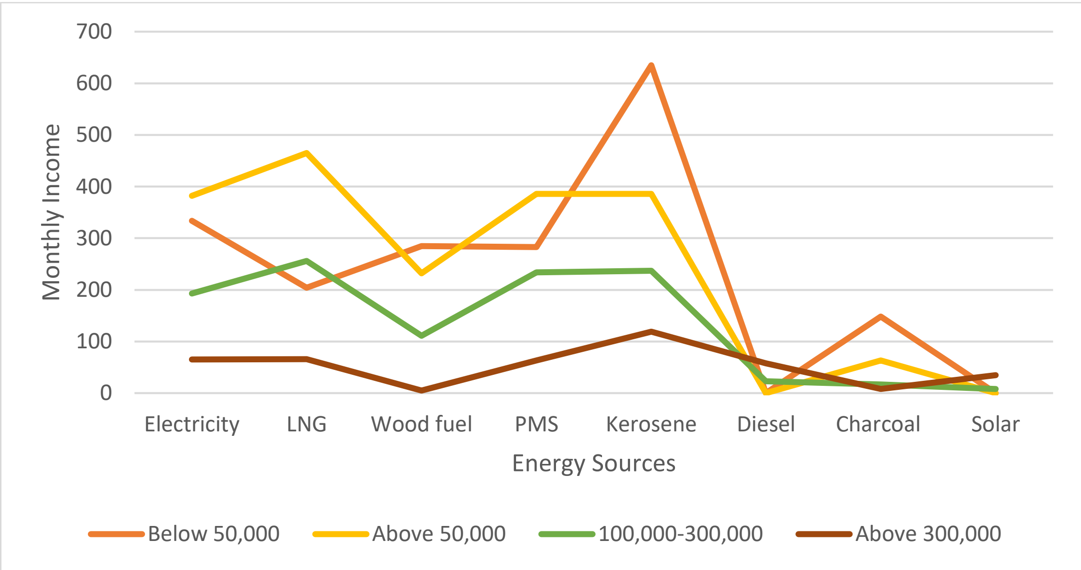

Fig. 1. *Graphical analysis of respondent income level in relation to their energy preference.*

4. Conclusion

The outcome of this study shown that kerosene is the most preferred energy sources in selected cities in Nigeria and this was basically due to the fact that it is not only relatively cheap but also affordable. Therefore, government must improve on policy to increase availability of these energy source. Also, poverty is a major factor that have alleviated sustainable energy. Thus, government should improve on the standard of living of its citizens. Renewable energy such as solar power and others should be encourage and maximized by the government by supporting local manufacturing of these equipment. This can be achieved through provision of credit grant from the government for indigenous technology.

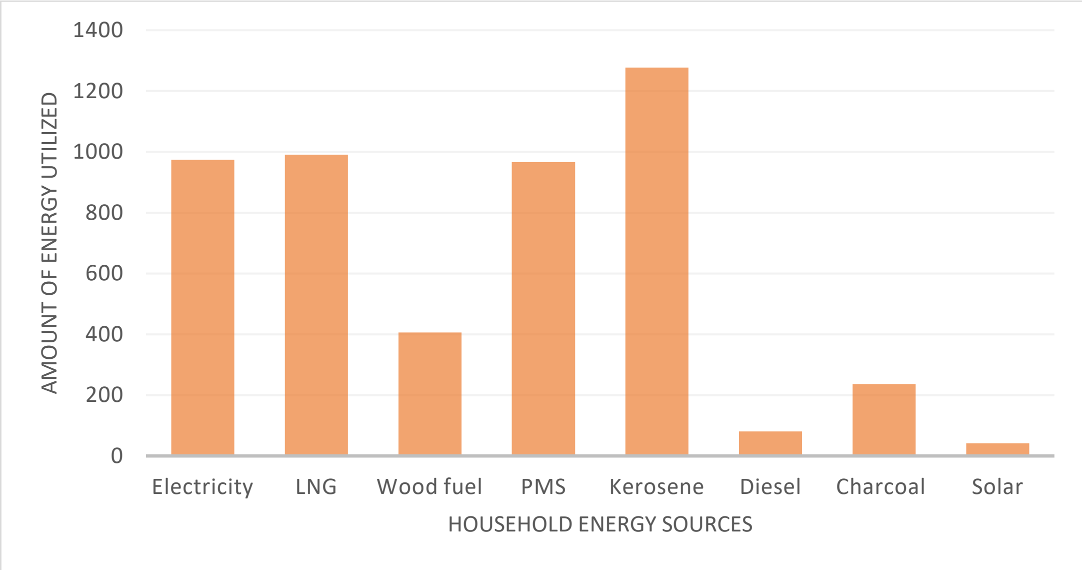

Fig. 2. *Respondent analysis on household energy utilization.*

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A Comprehensive Review on Meta-Heuristic Algorithms and their Classification with Novel Approach

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ABSTRACT

Conventional and classical optimization methods are not efficient enough to deal with complicated, NP-hard, high-dimensional, non-linear, and hybrid problems. In recent years, the application of meta-heuristic algorithms for such problems increased dramatically and it is widely used in various fields. These algorithms, in contrast to exact optimization methods, find the solutions which are very close to the global optimum solution as possible, in such a way that this solution satisfies the threshold constraint with an acceptable level. Most of the meta-heuristic algorithms are inspired by natural phenomena. In this research, a comprehensive review on meta-heuristic algorithms is presented to introduce a large number of them (i.e. about 110 algorithms). Moreover, this research provides a brief explanation along with the source of their inspiration for each algorithm. Also, these algorithms are categorized based on the type of algorithms (e.g. swarm-based, evolutionary, physics-based, and human-based), nature-inspired vs non-nature-inspired based, population-based vs single-solution based. Finally, we present a novel classification of meta-heuristic algorithms based on the country of origin.

1. Introduction

From the creation of human beings, they are constantly searching for perfection in many aspects of life. So, one of the most important concerns in the world is the search for optimal situations [1]. In the real world, many problems such as transportation, warehousing and product sales locations, communication networks design, scheduling, planning and etc. are all complicated and hybrid problems. These problems are practically large in such a way that they cannot be optimally solved within a reasonable and acceptable time. However, these problems need to be solved, so the alternative way is to accept the local or sub-optimal solutions with a suitable accuracy and optimization time. As a result, the heuristic algorithms are developed. They can be very effective and in some cases offer the global optimal

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solution. But one of their drawbacks is falling into local optimal traps without being able to get out of the situation. Therefore, meta-heuristic algorithms have been proposed to deal with this drawback [2]. Nowadays, new optimization methods have attracted many attentions and are under the spotlight in comparison with classical methods. Moreover, the application of new methods in complicated problems is increasing extensively [3].

Fig. 1. illustrates the optimization methods briefly [4] and [5].

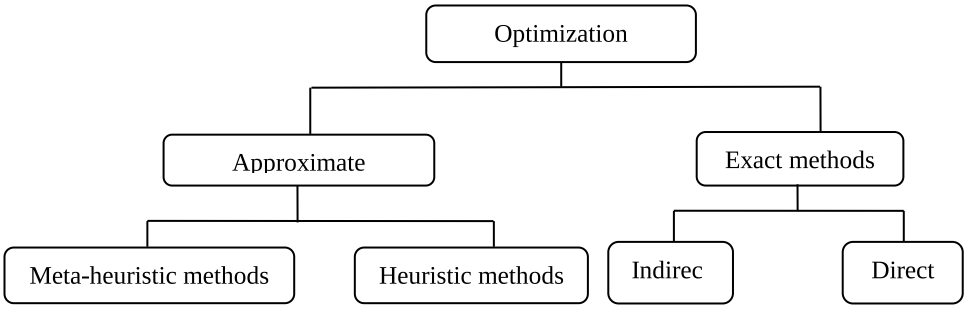

Fig. 1. *Optimization methods.*

Although, exact methods are very efficient in many problems [6]-[8], but they are not suitable for large-scale and non-linear problems. The more the size of the problem increases, the more the search space exponentially increases and it becomes impossible to search the solution space thoroughly [4]. Also, sometimes it is not possible to obtain the optimal solution with exact methods within a reasonable time due to the large scale and complexity, the scope of application of optimization in different fields, such as mathematics, computer science, engineering, economics and management. Such challenges have made the researcher to accept the local or sub-optimal solutions within a reasonable optimization time (i.e. approximate methods) [1]. In recent years, the application of meta-heuristic algorithms to solve NP-hard problems has been increased dramatically and is widely used in various fields. Unlike exact optimization methods, these approaches find the solutions that are very close to global optimal solution, in such a way that satisfies the threshold constraint with an acceptable level [9]. Also, these algorithms can solve such problems in an acceptable time [10]. Laporte and Osman [11] have defined meta-heuristic algorithm as "an iterative process that intelligently combines some heuristic concepts for searching and investigating in the whole solution space". Also, according to Voss et al. [12], a meta-heuristic algorithm is "A high-level iterative process that modifies and categorizes low-level heuristic solutions to generate high quality solutions. It may also manipulate a single or set of complete or incomplete solutions, iteratively". Combining meta-heuristic algorithms and their application is increasing these days. For examples: combinatorial optimization of permutation-based quadratic assignment problem using optics inspired optimization [13], factual power loss reduction by augmented monkey optimization algorithm [14] and a multi-level image thresholding approach based on the crow search algorithm and Otsu method [15].

Meta-heuristic algorithms which are a way to solve optimization problems start by generating random response (s) then move forward toward optimizing based on their operators and through changing the created random answers [16]. In general, all meta-heuristic algorithms use the similar mechanism to find the optimal solution. In most of these algorithms, the search starts by generating one or more random solutions in an acceptable range of variables. The primary generated solution in population-based algorithms is called population, colony, group, etc. and also each of solutions is called chromosome, particle, ant, and etc. Then, using operators and various methods of combining primary solutions, new solutions are generated. Moreover, the new solution will be chosen from the previous ones, and this process will continue until the stop criterion is met [17]. This process is illustrated in *Fig. 2*.

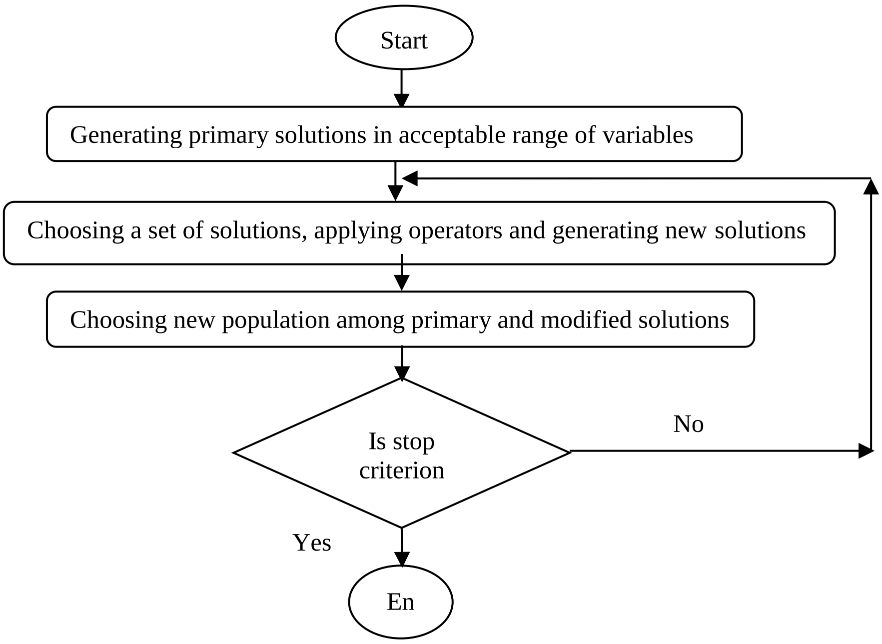

Fig. 2. *The overall process of meta-heuristic algorithms to find the optimal solution.*

In this research, a comprehensive review on meta-heuristic algorithms is presented to introduce a large number of them (i.e. about 110 algorithms). Also, this research provides a brief explanation along with the source of their inspiration for each algorithm. Moreover, these algorithms are categorized based on the type of algorithms (e.g. swarm based, evolutionary, physics based, and human based), nature-inspired vs. non-nature-inspired based, and finally we present a novel classification of meta-heuristic algorithms based on the country of origin. Accordingly, the rest of this paper is organized as follows: in Section 2, a comprehensive list of meta-heuristic algorithms with a brief explanation is provided. Section 3 discusses about the growth of meta-heuristic algorithm over time while Section 4 classifies the algorithms. Finally, the paper is concluded in Section 5.

2. A Comprehensive Introduction to 110 Meta-Heuristic Algorithms

A wide range of 110 meta-heuristic algorithms which have been developed in recent years, in order of presenting time and along with their developers and explanations, are provided in *Table 1*.

***Table 1**. A wide range of 110 meta-heuristic algorithms.*

Row	Algorithm	Developer(s)	Brief explanation	Year	Ref.
1	*Evolutionary Programming (EP)*	*Fogel & Fogel*	*This algorithm is one of the basic approaches for most modern and evolutionary methods that defines many operators such as mutations and crossover.*	*1965*	*[18]*
2	*Genetic Algorithm (GA)*	*Holland*	*This algorithm is a searching method for finding the approximate solution of optimization problems using biological concepts such as inheritance and mutation. This algorithm is one of the most popular population-based, heuristic algorithms. The main operator of this algorithm is combination, but the mutation operator is also used to prevent unacceptable convergence and falling into the local optimal traps. This algorithm is based on the Darwin's theory and the principle of survival of the fittest. Also, the basic idea is to inherit traits by genes.*	*1975*	*[19]*
3	*Scatter Search Algorithm (SSA)*	*Glover*	*This algorithm is different from other evolutionary algorithms. It is based on the concept of systematic approaches to generate new solutions. This systematic approach has some advantages in comparison with purely randomized choosing solutions. Systematic approaches are used for diversification and intensification in searching process.*	*1977*	*[20]*
4	*Simulated Annealing (SA)*	*Kirkpatrick et al.*	*This algorithm is a probability-based method for finding the global optimal solution in problems with large solution space. Also, it is single-solution (different from most of meta-heuristic algorithms). SA is proposed based on the process of melting and freezing metals on the molecular scale. This process requires heating and then cooling a material gradually, in order to obtain a strong and solid crystal structure.*	*1983*	*[21]*
5	*Tabu Search (TS)*	*Glover*	*This algorithm works almost like local search algorithms, except that it uses a concept called Tabu List to avoid falling into the local optimal traps. The algorithm starts with an initial solution and searches for the neighborhood around it and chooses the best one and moves to that point under some conditions. Moving from current solution to candidate neighbor solution is allowed when it is not on the tabu list. Otherwise, the next neighbor solution, which is ranked next in the evaluation of neighbor solutions, will be chosen. The length of the tabu list indicates the maximum number of tabu iterations in the search process.*	*1986*	*[22]*

Row	Algorithm	Developer(s)	Brief explanation	Year	Ref.
6	*Cultural Algorithms (CA)*	*Reynolds*	*This algorithm is a category of evolutionary algorithms which in comparison with other algorithm and along with population component, has knowledge component too. There are various classifications of belief space called temporal knowledge, domain specific knowledge, situational knowledge, and spatial knowledge.*	*1994*	*[23]*
7	*Particle Swarm Optimization (PSO)*	*Kennedy & Eberhart*	*The algorithm is inspired by group flight of birds. Each particle calculates the value of the objective function in a position of solution space. Then, for each particle, by combining the information of current location and the best location it previously had, as well as the information of one or more of the best particles in the group, it chooses the direction to move. After moving all particles, one step finishes. These steps are repeated several times to obtain the desired result i.e. when the stop criterion is met. This algorithm has two operators: speed updating and position updating operators.*	*1995*	*[24]*
8	*Ant Colony Optimization (ACO)*	*Dorigo et al.*	*ACO is inspired by nature which explores the behavior of real ants. This algorithm is one of the population-based meta-heuristic algorithms. Researchers have shown that ants are social animals that live in colonies, and their behavior is more about the survival of colony than about the survival of a component. One of the most interesting and important behavior of ants is their approaches in finding food, and in particular how to find the shortest route between food resources and nests. Communication between ants with each other or between ants and the environment is based on the use of a chemical called pheromones. The ants leave a pheromone trace of themselves while walking. Although this material evaporates rapidly, but in the short term it remains as the ant's track on the surface of the earth. When ants want to choose between two paths, they usually choose a path that has more pheromones.*	*1996*	*[25]*
9	*Differential Evolution (DE)*	*Storn & Price*	*This algorithm is based on the theory of natural evolution and, similar to genetic algorithm, is based on mutation and combination operators. Although this algorithm uses mutation and combination operators but their utilization is different from GA. Moreover, in the step of comparison between new and old population and also choosing the best solution, this algorithm operates different from GA.*	*1997*	*[26]*

Row	Algorithm	Developer(s)	Brief explanation	Year	Ref.
10	*Variable Neighborhood Search (VNS)*	*Mladenović & Hansen*	*This algorithm is proposed based on the rules of simple local search. In this algorithm, the solution space is divided into several neighborhoods, and then one of these neighbors is randomly selected and local search is performed. If the solution is better than the best one recorded, a new neighborhood will be generated around the solution.*	*1997*	*[27]*
11	*Sheep Flocks Heredity Model (SFHM)*	*Kim & Ahn*	*This algorithm is an evolutionary computation algorithm based on sheep flocks heredity. It simulates heredity of sheep flocks in a prairie.*	*2001*	*[28]*
12	*Harmony Search (HS)*	*Geem et al.*	*Harmony is a relationship between different sound waves with different frequencies. The best harmony provides the best aesthetic experience for the audiences. This algorithm traces the similarity between finding a harmony in a music performance and finding the optimal solution for an optimization problem.*	*2001*	*[29]*
13	*Bacterial Foraging Optimization (BFO)*	*Passino*	*The algorithm is inspired by the bacterial motility process for finding food resources. Individual bacterial search behavior, probability of reproduction regarding the fact that reproduction is only for those bacteria that are well fed and finally, bacterial decomposition are the key elements of this algorithm.*	*2002*	*[30]*
14	*Social Cognitive Optimization (SCO)*	*Xie et al.*	*This algorithm is based on the development of knowledge and intelligence in humans and uses the concept that people learn by seeing others and the consequences of their behavior. It also considers personal, behavioral, and environmental factors.*	*2002*	*[31]*
15	*Shuffled Frog Leaping Algorithm (SFLA)*	*Eusuff & Lansey*	*This algorithm combines deterministic and probabilistic approaches. The deterministic aspects allow the algorithm to efficiently use the solution information to lead the evolutionary approach while probabilistic aspects guarantee the flexibility of algorithm. The base of SFL is to simulate the behavior of frogs to find food in wetlands. The algorithm tries to make a balance between extensive investigation in a solution space and searching around possible solutions. In this algorithm, population consists of a set of frogs (i.e. solutions) and each frog has a chromosome-like structure similar to genetic algorithm.*	*2003*	*[32]*
16	*Electromagnetism-like algorithm (EMA)*	*Birbil & Fang*	*This algorithm is proposed based on electrostatic system rules. In this algorithm, each particle has a virtual electric charge in such a way that the amount of this charge is related to the optimality of the point at which the particle is located.*	*2003*	*[33]*

Row	Algorithm	Developer(s)	Brief explanation	Year	Ref.
17	*Space Gravitational Algorithm (SGA)*	*Hsiao et al.*	*The algorithm is inspired by the simulation of several asteroids in space and they are constantly moving to find the heaviest ones. The SGA utilizes Einstein's theory of relativity and Newton's law of gravitation to search the global optimal solution. Asteroids change their position independently, so the computational complexity decreases and the probability of falling into local optimal traps is very low.*	*2005*	*[34]*
18	*Particle Collision Algorithm (PCA)*	*Sacco & Oliveira*	*This algorithm is inspired by the nuclear collision reactions, especially scattering and absorption. The PCA structure is similar to SA but does not rely on user-defined parameters and does not require a cooling plan.*	*2005*	*[35]*
19	*Big Bang-Big Crunch (BB-BC)*	*Erol & Eksin*	*This algorithm is proposed based on the Big Bang theory and freezing the universe. In BB-BC, a weighted center of gravity is firstly constructed using each of the solutions and their fitness. Then, new solutions are generated in the neighborhood of this center, using the Gaussian distribution.*	*2006*	*[36]*
20	*Group Search Optimizer (GSO)*	*He et al.*	*This algorithm is inspired by the behavior of group of animals in the search for food resources. In this algorithm, the group is divided into three types namely producers, scrounger and rangers, and each type has its specific behavior.*	*2006*	*[37]*
21	*Invasive Weed Optimization (IWO)*	*Mehrabian & Lucas*	*This algorithm is inspired by weed colonies. Weed colonies are strong enough to pose a threat to useful plants. Weeds are also adaptable to environmental changes. Being strong, random, and adaptable is modeled as a numerical optimization method.*	*2006*	*[38]*
22	*Small-world Optimization Algorithm (SWOA)*	*Du et al.*	*This algorithm is proposed based on some scientific experiments on human communication and networking approaches. This method uses local short-range and random long-range search agents for local and global search, respectively.*	*2006*	*[39]*
23	*Cat Swarm Optimization (CSO)*	*Shu et al.*	*This algorithm is based on the behavior of cats and has two sub-models, namely: tracking and searching. In this algorithm, cats are considered as the number of solutions. Each cat has a relative velocity for on each dimension, a proportional value, and a parameter indicating the status.*	*2006*	*[40]*
24	*Saplings Growing UP Algorithm (SGA)*	*Karci & Alatas.*	*This algorithm is inspired by the planting and growth of saplings, which has two stages: the planting stage and the growth stage. Uniform planting is done with the aim of uniformly distributing the searching agents in the solution space. The growth stage consists of three operators: mating, branching, and vaccination.*	*2006*	*[41]*

Row	Algorithm	Developer(s)	Brief explanation	Year	Ref.
25	*Imperialist Competitive Algorithm (ICA)*	*Atashpaz-Gargari & Lucas*	*In this algorithm, members are considered as countries. Countries are divided into colonial and colonizer groups and empires are built. During the iterations, the weak empires collapse and the strong empire takes control of its colonies. Convergence occurs when there is only one empire left and all the other colonies are equal.*	*2007*	*[42]*
26	*Artificial Bee Colony Algorithm (ABC)*	*Karaboga & Basturk*	*In this algorithm, bee group behavior is used to find the food resources. The performance of worker bees, watchdogs and scouts are modeled. The equivalent to these three types of bees, three operators are considered namely deterministic obtaining information from neighborhood, probabilistic obtaining information from neighborhood and the searching for new areas if no improvement is achieved.*	*2007*	*[43]*
27	*Central Force Optimization (CFO)*	*Formato*	*Unlike many stochastic algorithms, CFO is a deterministic method and do not need to generate random solutions. In this algorithm, searching process moves under the influence of gravity in the decision space and changes the position of agents according to equations and constraints. Therefore, during the searching process, the solution moves slowly towards the searches with highest proportion or mass.*	*2007*	*[44]*
28	*Integrated Radiation Algorithm (IRA)*	*Chuang & Jiang*	*This algorithm is based on the concept of gravitational radiation in Einstein's theory of general relativity. This important theory is used to search for the optimal solution in the solution space. In IRA, the search space is regarded as a simplified astrophysics that contains search agents, and these agents are randomly distributed in the search space. It is assumed that the solution which finds the best objective function has a supernova with an incomplete symmetrically expanding shape.*	*2007*	*[45]*
29	*Multi Point Simulated Annealing Algorithm (MPSA)*	*Lamberti & Pappalettere*	*This algorithm utilizes a multi-level simulated annealing scheme where different candidate designs are compared simultaneously.*	*2007*	*[46]*
30	*River Formation Dynamics Algorithm (RFDA)*	*Rabanal et al.*	*This algorithm is inspired by how the river and its bed are formed using erosion and sedimentation process. Some important factors such as moving from high to low, erosion and sedimentation are considered in this model.*	*2007*	*[47]*
31	*Big Crunch Algorithm (BCA)*	*Kripka & Kripka*	*This algorithm is based on closed world theory. The kinetic energy generated by the first cosmic explosion (i.e. the Big Bang) overcomes the gravitational energy of components. As the beginning of universe, this explosion will be done with infinite heat and energy. This process will continue as long as only one component (i.e. mass) remains in the world and this leads to an acceptable result.*	*2008*	*[48]*

Row	Algorithm	Developer(s)	Brief explanation	Year	Ref.
32	*Biogeography Based Optimization (BBO)*	*Simon*	*This algorithm is inspired by the geographical distribution of biological organisms and uses the basic operators such as migration and mutation.*	*2008*	*[49]*
33	*Firefly Algorithm (FFA)*	*Yang*	*The algorithm is inspired by the illumination of firefly insects to mate, hunt, and scare the enemies. In FFA, insects with the ability to generate more light attract the weaker insects and it relates inversely to the distance between them.*	*2009*	*[50]*
34	*Paddy Field Algorithm (PFA)*	*Premaratne et al.*	*This algorithm starts with randomly spreading the seeds (i.e. search agents). After a while and turning seeds into plants, those with higher growth (i.e. better fitness) are more likely to be re-used. All plants spread their seeds to overcome local optimal traps.*	*2009*	*[51]*
35	*Gravitational Search Algorithm (GSA)*	*Rashedi et al.*	*This algorithm is based on the laws of gravity and motion. According to the law of gravity, each particle in the universe attracts another particle with a force that is proportional to the mass and inversely proportional to the square of the distance between the particles. In this algorithm, each agent is considered as an object and the performance of these objects is measured by their mass. Therefore, it is expected that at the end of optimization process the position of object with the heaviest mass shows the optimal global solution.*	*2009*	*[52]*
36	*Cuckoo Search (CS)*	*Yang & Deb*	*This algorithm is proposed based on the cuckoo behavior in laying nests of other birds. The final aim is to maximize the production of chickens while these eggs not to be identified by the host bird.*	*2009*	*[53]*
37	*Hunting Search (HuS)*	*Oftadeh & Mahjoob*	*This algorithm is inspired by the group hunting of animals such as lions, wolves and dolphins. The algorithm indicates that although the hunting methods of these animals are different, but they share the same approach for hunting and chasing the prey. In other words, hunters encircle the prey and gradually tighten the siege to catch it. In addition, each member of the group adjusts its position according to the position of its own and other members. If the prey escapes the siege, the hunters reorganize the group to re-siege the prey.*	*2009*	*[54]*
38	*Intelligent Water Drops (IWD)*	*Shah-Hosseini*	*The algorithm is inspired by the flow of water along the path. As the water flows between two points, the water speed, the amount of soil with the water, and the soil bed change. In this algorithm, these changes are modeled, accordingly. In other words, intelligent water drops are used as search agents that work together to find the optimal solution.*	*2009*	*[55]*
39	*Artificial Physics Optimization Algorithm (APOA)*	*Xie et al.*	*This algorithm is inspired by physical forces. In APA, each agent is regarded as a physical particle that has a certain mass, velocity and position. Virtual forces move these particles to areas with better fitness. The amount of fitness depends on the user defined agent mass. With these physical rules, agents search the solution space.*	*2009*	*[56]*

Row	Algorithm	Developer(s)	Brief explanation	Year	Ref.
40	*Bacterial Evolutionary Algorithm (BEA)*	*Das et al.*	*This algorithm is an evolutionary clustering method to classify datasets for optimal number of groups. The algorithm is inspired by microbial evolution and utilizes two specific operators: bacterial mutation and gene transfer operations.*	*2009*	*[57]*
41	*Human-inspired Algorithm (HIA)*	*Zhang et al.*	*The algorithm mimics search methods according to climbers who use modern facilities such as binoculars and cell phones to find the highest mountain peak. The interesting feature of this method is that it divides the search space evenly into sub-spaces and allocates an equal number of search agents to them.*	*2009*	*[58]*
42	*League Championship Algorithm (LCA)*	*Kashan*	*This algorithm is based on the competition between teams in a sports league. Search agents are considered as teams that compete for several weeks (i.e. iterations). The competition is between two competitors and the fittest is considered as winner. At the end of each iteration, all teams are ready to make changes for the next week.*	*2009*	*[59]*
43	*Locust Swarms (LS)*	*Chen*	*This algorithm is inspired by the swarm of locusts. The algorithm starts with intelligent starting points. Then, the PSO method and greedy local search algorithm are used to explore the search space. Search agents start with a little distance from the previous solution.*	*2009*	*[60]*
44	*Consultant-Guided Search (CGS)*	*Iordache*	*This algorithm is based on the direct exchange of information between individuals within a population. CGS is a collective intelligence technique inspired by the real-world decision making approaches.*	*2010*	*[61]*
45	*Bat Algorithm (BA)*	*Yang*	*This algorithm is inspired by the bat sound system for detecting the prey, barrier and nest locations in the darkness. In BA, each bat changes its speed and position according to the best existing positions.*	*2010*	*[62]*
46	*Charged System Search (CSS)*	*Kaveh & Talatahari*	*In this algorithm, each agent is considered as a charged particle (CP) and is a candidate for solution. The law of motion is also used to guide the CP movements. Each CP according to the distance and value of the objective function (i.e. its charge value), is affected by other CPs and the force applied to each CP determines the new position, velocity and acceleration.*	*2010*	*[63]*
47	*Chemical Reaction Optimization (CRO)*	*Lam & Li*	*The algorithm is inspired by the behavior of molecules in chemical reactions and the exchange of energy between them. By modeling the chemical combination and decomposition reactions, the molecules behave in such a way that they can minimize their potential energy (i.e. cost function).*	*2009*	*[64]*
48	*Eagle Strategy Algorithm (ESA)*	*Yang & Deb*	*This algorithm is a two-step hybrid search method that combines random search with firefly algorithm.*	*2010*	*[65]*
49	*Group Counseling Optimization (GCO)*	*Eita & Fahmy*	*This algorithm mimics human problem solving behavior through consultation. Iterations in this algorithm are considered as counseling sessions. In these sessions members constantly improve their position with the help of themselves or the counseling team.*	*2010*	*[66]*

Row	Algorithm	Developer(s)	Brief explanation	Year	Ref.
50	*Social Emotional Optimization (SEO)*	*Xu et al.*	*This algorithm uses the concept of human efforts to gain higher social status. Members of a population are treated as individual persons in a society. Each person tries to increase his /her emotional score. Based on this emotional index and feedback from other members of the society, social status becomes updated. Finally, members with the highest social status are the optimal solution.*	*2010*	*[67]*
51	*Galaxy Based Search Algorithm (GbSA)*	*Shah-Hosseini*	*This algorithm is inspired by the spiral arm of galaxies to approach optimal solution. This means that GbSA searches the solution space for better solutions using these spiral-like arms. To escape the local optimal traps, spiral movements are improved by chaos.*	*2011*	*[68], [69]*
52	*Spiral Dynamics Inspired Optimization (SDIO)*	*Tamura & Yasuda*	*This algorithm is inspired by spiral phenomenon in the nature. This spiral movement is observed in many structures, such as galaxies, tornadoes and vortices. The algorithm uses a multidimensional spiral to search the solution space. It also has several control parameters to balance the variation and intensification.*	*2011*	*[70]*
53	*Teaching-learning based Optimization (TLBO)*	*Rao et al.*	*This algorithm is based on the mutual teaching-learning relationship, while the members of population are treated as students of a class. The searching process consists of two phases: in the first phase the learning process proceeds by teacher's influence, while in the second phase it is done by mutual interactions.*	*2011*	*[71]*
54	*Anarchic Society Optimization (ASO)*	*Shayeghi & Dadashpour*	*The algorithm simulates the problem as a group with abnormal, unstable and disruptive members to overcome local optimal traps.*	*2012*	*[72]*
55	*Current Search (CS)*	*Sakulin & Puangdownreong*	*This algorithm is based on the electrical flow behavior in electric circuits. Usually, current chooses the less resistive path among other paths.*	*2012*	*[73]*
56	*Water Cycle Algorithm (WCA)*	*Eskandar et al.*	*This algorithm is inspired by the nature of water cycle process. The solutions in this algorithm are: creek, river (i.e. some of the best solutions) and sea as the best solution. The algorithm is implemented by moving creeks toward the river and rivers toward the sea. If the creeks have enough qualification they can join the river or even the sea.*	*2012*	*[74]*
57	*Wolf Search Algorithm (WSA)*	*Tang et al.*	*The algorithm is inspired by the life of wolves and how they search for food and survive from threats. In this algorithm, each agent searches independently while keeps the previous position in its memory. If the new position is the best among all previous ones, then algorithm integrates the current agent with another agent.*	*2012*	*[75]*
58	*Mine Blast Algorithm (MBA)*	*Sadollah et al.*	*This algorithm is proposed based on a real world mine-blast event. In an explosion a number of explosive bullets are fired. Each piece of explosive may generate another new explosion. The explosive component that causes the most damages is chosen to expand the new mine.*	*2012*	*[76]*

Row	Algorithm	Developer(s)	Brief explanation	Year	Ref.
59	*Atmosphere Clouds Model (ACM)*	*Yan & Hao*	*This algorithm mimics the behaviors of a cloud in the natural world. It simulates the generation, move and spread behavior of a cloud for intensification and diversification / exploration in the search space.*	*2013*	*[77]*
60	*Black Holes Algorithm (BHA)*	*Hatamlou*	*The algorithm is inspired by the black hole phenomenon. In this algorithm, the best solution in each iteration is considered as a black hole that other stars are attracted to it. When a star becomes very close to the black hole, they merge and a new star (i.e. solution) creates randomly.*	*2013*	*[78]*
61	*Egyptian Vulture Optimization (EVO)*	*Sur et al.*	*The algorithm is inspired by the natural behaviors of Egyptian vultures and how they search for food resources. This algorithm was originally designed for hybrid optimization problems.*	*2013*	*[79]*
62	*Penguins Search Optimization Algorithm (PSOA)*	*Gheraibia & Moussaoui*	*The basic idea of this algorithm is the group behavior of penguins when they go for hunting. Each penguin starts the local search process while he is aware of its own location and the food that other members were found. The group with the most food (i.e. fishes) is selected as the best solution.*	*2013*	*[80]*
63	*Swallow Swarm optimization (SSO)*	*Neshat et al.*	*This algorithm is proposed from the behavior of swarm of swallows and their movements. The particles in this algorithm are divided into three categories: explorer or discoverer, aimless and leader.*	*2013*	*[81]*
64	*Grey Wolf Optimizer (GWO)*	*Mirjalili et al.*	*This algorithm is inspired by gray wolves and mimics the predation process and hierarchy of gray wolves. In GWO, four types of gray wolves, namely Alpha, Beta, Delta and Omega, are introduced to simulate leadership hierarchies. In addition, three major hunting steps called searching for prey, sieging the prey and attacking to prey are considered in this method.*	*2014*	*[82]*
65	*Golden Ball (GB)*	*Osaba et al.*	*This algorithm is based on a multiple population approach and relies on the concepts of football game.*	*2014*	*[83]*
66	*Animal Migration Optimization Algorithm (AMOA)*	*Li et al.*	*This algorithm is inspired by the group migration of animals and how they leave one group to survive in other ones.*	*2014*	*[84]*
67	*Soccer League Competition Algorithm (SLC)*	*Moosavian & Roodsari*	*The theory behind this approach is inspired by soccer leagues and is based on the competition between teams and players. Competition between teams for better ranking in the league and internal competition between players for personal development will result in convergence towards global optimality. The simulation results of applying the SLC to nonlinear systems verifies that this algorithm in comparison to other meta-heuristic algorithms converges to solution faster and more accurately.*	*2014*	*[85], [86]*
68	*Chicken Swarm (CS)*	*Meng et al.*	*This algorithm simulates the hierarchical behavior of a group of chickens and consists of: roosters, hens and chickens. Chickens are divided into several groups. Each group has a rooster and a large number of hens and chickens and they compete in a particular hierarchical order.*	*2014*	*[87]*

Row	Algorithm	Developer(s)	Brief explanation	Year	Ref.
69	*Forest Optimization Algorithm (FOA)*	*Ghaemi & Feizi-Derakhshi*	*This algorithm is based on the process of sowing seeds in a forest. In this algorithm, the model assumes that the seeds right under the trees cannot grow, but the seeds scattered elsewhere may grow and become another trees.*	*2014*	*[88]*
70	*Heart Algorithm (HA)*	*Hatamlou*	*This algorithm models the optimization problem as the heart and circulatory system. In this method, a member of population that has the best fits is considered as the heart and other members are considered as blood molecules. Blood molecules move toward or against the heart to gain better fitness.*	*2014*	*[89]*
71	*Kaizen Programming (KP)*	*De Melo*	*Kaizen is a Japanese approach to solve problems. Unlike evolutionary approaches which each agent is a complete solution, in this method each expert proposes an idea to solve the problem and the final solution combines all ideas together. The fitness of each idea is measured by its contribution to find the final solution.*	*2014*	*[90]*
72	*Exchange Market Algorithm (EMA)*	*Ghorbani & Babaei*	*This method as an evolutionary algorithm is inspired by the procedure of trading the shares on stock market.*	*2014*	*[91]*
73	*African Buffalo Optimization (ABO)*	*Odili et al.*	*This algorithm draws its inspiration from the behavior of African buffalos in the vast African forests and savannahs. African buffalos are a wild species of domestic cattle and are always mobile tracking the rainy seasons in different parts of Africa in search of lush green pastures to satisfy their large appetites.*	*2015*	*[92]*
74	*Elephant Herding Optimization (EHO)*	*Wang at el.*	*This algorithm is a kind of swarm-based meta-heuristic search method, and is inspired by the herding behavior of elephant group.*	*2015*	*[93]*
75	*Ions Motion Algorithm (IMA)*	*Javidy at el.*	*This algorithm uses the ionic movements and the tensile force and pressure of anions and cations. In this algorithm, the candidate solutions are divided into two groups: 1-anions (negative ions) and 2-cations (positive ions). Ions introduce the candidate solutions for a particular problem and tensile /pressure forces allow the ions to move in the search space.*	*2015*	*[94]*
76	*General Relativity Search Algorithm (GRSA)*	*Beiranvand & Rokrok*	*This algorithm is based on the concept of general theory of relativity. In this algorithm, members of population are modeled as particles in space in such a way that they are not affected by any forces except gravity. These particles are designed to reach to their most stable situation and move in the shortest possible paths. Length of steps and directions are calculated according to speed and shortest paths.*	*2015*	*[95]*
77	*Jaguar Algorithm with Learning Behavior (JALB)*	*Chen et al.*	*This algorithm is inspired by the jaguar's hunting behavior. A jaguar aims to catch a prey and moves rapidly toward the target. Jaguars are also used to go for hunting as a team. This algorithm mimics the jaguar's hunting behavior to balance exploration and exploitation.*	*2015*	*[96]*
78	*Optics Inspired Optimization (OIO)*	*Kashan*	*A concave mirror converges light beams while a convex mirror diverges light beams. In this algorithm, the optics phenomenon is modeled as an optimizer. This algorithm considers the search space as a reflective mirror in such a way that each peak and valley acts as a convex mirror and concave mirror, respectively.*	*2015*	*[97]*

Row	Algorithm	Developer(s)	Brief explanation	Year	Ref.
79	*Runner-Root Algorithm (RRA)*	*Merrikh-Bayat*	*This algorithm models the goals of roots and runners in plants. Runners search for a large area with large steps while roots search for a small area. The algorithm also has two functions corresponding to runners and roots for exploration and exploitation, respectively.*	*2015*	*[98]*
80	*Vortex Search Algorithm (VSA)*	*Doğan & Ölmez.*	*This method is one of meta-heuristic algorithm which is based on the vertical flow pattern of fluids. This algorithm uses an adaptive step length mechanism to exploit and explore the searches.*	*2015*	*[99]*
81	*Stochastic Fractal Search (SFS)*	*Salimi*	*The algorithm is inspired by the natural phenomenon of growth that uses a mathematical concept called fractal.*	*2015*	*[100]*
82	*Prey-Predator Algorithm (PPA)*	*Tilahun & Ong*	*The algorithm is inspired by the predator-prey relationship in animals.*	*2015*	*[101]*
83	*Water Wave Optimization (WWO)*	*Zheng*	*This algorithm is inspired by the theory of water waves. It is verified that some phenomena which have the characteristics of water waves such as propagation, reflection, and breaking, can be used to develop an effective mechanisms for searching in a high-dimensional solution space.*	*2015*	*[102]*
84	*Bull Optimization Algorithm (BOA)*	*Findik*	*This algorithm modifies the selection process in the genetic algorithm in such a way that only better members can participate in crossover.*	*2015*	*[103]*
85	*Elephant Search Algorithm (ESA)*	*Deb et al.*	*In this algorithm, members of a population are considered as elephants in a herd. Male members and female ones are considered as exploring searching agents and local searching agent, respectively.*	*2015*	*[104]*
86	*Ant Lion Optimizer (ALO)*	*Mirjalili*	*The algorithm is inspired by ant lion hunting behaviors and is based on the following steps: random search, trapping, sieging, catching the prey, and trap reconstruction.*	*2015*	*[105]*
87	*Lion Optimization Algorithm (LOA)*	*Yazdani & Jolai*	*This algorithm is inspired by the social behavior of lions. Lions live in both resident and nomad types. Adult male lions freely leave the group and move around. Resident lions are modeled as local search agents, while nomad lions are modeled as global search agents for identification and exploration.*	*2016*	*[106]*
88	*Whale Optimization Algorithm (WOA)*	*Mirjalili & Lewis.*	*This algorithm is inspired by nature and mimics the social behavior of whales. This algorithm has three main steps: sieging the prey, attacking to prey (i.e. exploitation step) and searching for prey (i.e. exploration step).*	*2016*	*[107]*
89	*Dynamic Virtual Bats Algorithm (DVBA)*	*Topal & Altun*	*This algorithm is inspired by bat's ability to generate various wavelengths and frequencies during hunting process.*	*2016*	*[108]*
90	*Tug of War Optimization (TWO)*	*Kaveh & Zolghadr*	*This algorithm is inspired by tug of war game and is based on population. In this algorithm, each candidate solution is considered as a team to play the game.*	*2016*	*[109]*
91	*Virus Optimization Algorithm (VOA)*	*Liang & Cuevas Juarez*	*This algorithm mimics the behavior of viruses in attacking cells. In this method, the number of viruses in each iteration is controlled by the immune system to prevent unwanted growth of virus population.*	*2016*	*[110]*

Row	Algorithm	Developer(s)	Brief explanation	Year	Ref.
92	*Virus colony search (VCS)*	*Li et al.*	*This algorithm mimics the strategy of virus replication and propagation in infecting host cells.*	*2016*	*[111]*
93	*Crow Search Algorithm (CSA)*	*Askarzadeh*	*This algorithm is inspired by the intelligent behavior of crows and also is a population-based technique. The basic CSA idea is that crows hide and store their surplus food and find it when needed.*	*2016*	*[112]*
94	*Dragonfly Algorithm (DA)*	*Mirjalili*	*This algorithm is inspired by the social behavior of dragonflies when they are searching for food, guiding the group and escaping from enemy.*	*2016*	*[113]*
95	*Camel Algorithm (CA)*	*Ibrahim & Ali*	*This algorithm is inspired by the behavior of camel during a desert march. This method considers the factors such as temperature, water supply, stability, visibility, and ground conditions.*	*2016*	*[114]*
96	*Water Evaporation Optimization (WEO)*	*Kaveh & Bakhshpoori*	*This algorithm is based on the behavior of evaporation of water molecules from solid surfaces with different characteristics. Water molecules are considered as the members of population and the solid surface is considered as the solution space. Surface wettability with other molecular properties is regarded as the search parameters.*	*2016*	*[115]*
97	*Thermal Exchange Optimization (TEO)*	*Kaveh & Dadras*	*This algorithm is based on the Newton's law of cooling, which is simple and straightforward.*	*2017*	*[116]*
98	*Electro-Search Algorithm (ESA)*	*Tabari & Arshad*	*This algorithm is inspired by the movement of electrons around the nucleus of atoms.*	*2017*	*[117]*
99	*Grasshopper Optimisation Algorithm (GOA)*	*Saremi et al.*	*This algorithm models the optimization problems as a group of grasshoppers. In this method the neonatal stage is modeled the same as adult stage. Young grasshoppers move slowly with small steps, while adults move suddenly with bigger steps.*	*2017*	*[118]*
100	*Sperm Motility Algorithm (SMA)*	*Raouf & Hezam*	*This algorithm is based on the human reproductive system. Search agents (i.e. sperms) spread in solution space randomly. In this method, the random sperm motility is modeled to search for ovum. The chemical secretion of ovum attracts the sperm to the optimal solution.*	*2017*	*[119]*
101	*Beetle Swarm Optimization Algorithm (BSOA)*	*Wang & Yang*	*This algorithm is proposed by enhancing the performance of swarm optimization through beetle foraging principles.*	*2018*	*[120]*
102	*Chaotic Bird Swarm Optimization Algorithm*	*Ismail et al.*	*This algorithm combines the chaotic-based methods with foraging and privilege behaviors to improve exploitation quality.*	*2018*	*[121]*
103	*Butterfly Optimization Algorithm*	*Arora & Singh*	*This algorithm mimics food search and mating behavior of butterflies, to solve global optimization.*	*2019*	*[122]*
104	*Chaotic grasshopper Optimization Algorithm (CGOA)*	*Arora & Anand*	*This method introduces chaos theory into the optimization process of GOA. The chaotic maps balance the exploration and exploitation efficiently.*	*2019*	*[123]*

Row	Algorithm	Developer(s)	Brief explanation	Year	Ref.
105	*Quantum Dolphin Swarm Algorithm*	*Qiao & Yang*	*In this method, quantum search algorithm is introduced into Dolphin Swarm Algorithm to escape the local optimum.*	*2019*	*[124]*
106	*Emperor Penguins Colony*	*Harifi et al.*	*This algorithm, inspired by the behavior of Emperor Penguins. It is controlled by the body heat radiation of the Penguins and their spiral-like movement in their colony.*	*2019*	*[125]*
107	*Shell Game Optimization*	*Dehghani et al.*	*This method simulates the rules of a game known as shell game to design an algorithm for solving optimization problems.*	*2020*	*[126]*
108	*Darts Game Optimizer (DGO)*	*Dehghani et al.*	*This method simulates the rules of Darts game to design an algorithm for solving optimization problems.*	*2020*	*[127]*
109	*Capuchin Search Algorithm (CapSA)*	*Braik et al.*	*This algorithm is inspired by the dynamic behavior of capuchin monkeys.*	*2020*	*[128]*
110	*Red deer Algorithm*	*Fathollahi-Fard et al.*	*This algorithm mimics the behavior of Scottish red deer. Its main inspiration originates from an unusual mating behavior of Scottish red deer in a breading season.*	*2020*	*[129]*

3. Growth and Expansion of Meta-Heuristic Algorithms

Most of meta-heuristic algorithms have been developed in recent years (i.e. after 2000). The growth and development process of these algorithms over time is illustrated in *Fig. 3*.

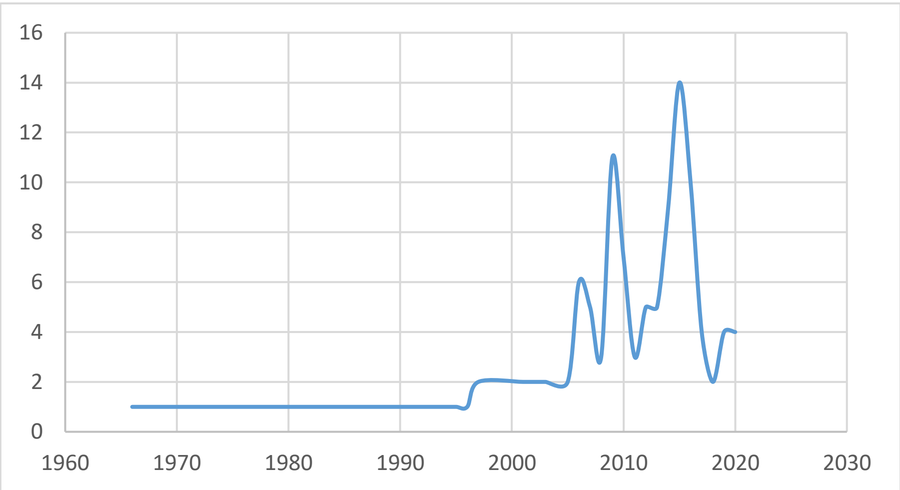

Fig. 3. *Growth and development process of meta-heuristic algorithms over time.*

4. Classification of Meta-Heuristic Algorithms

According to the features of each algorithm, different classifications are introduced. In this section, the meta-heuristic algorithms are studied and compared from five aspects. Then classifications of algorithms based on these aspects are presented [4], [91], [103], [130], and [131]. These classifications are according to:

- The type of algorithms (swarm-based, evolutionary-based, physics-based and human-based).
- Nature-inspired vs. non-nature-inspired.
- The source of inspiration.
- Population based vs. single solution based.
- Based on the country of origin.

The classification of 110 investigated meta-heuristic algorithms based on the type of algorithms (swarm-based, evolutionary-based, physics-based, human-based), nature-inspired vs. non-nature-inspired, the source of inspiration and population based vs. single solution based is illustrated in *Fig.4*. While *Fig. 5, Fig. 6, Fig. 7* and *Fig. 8* show the above categories statistically.

The classifications of 110 investigated meta-heuristic algorithm based on the country of origin is

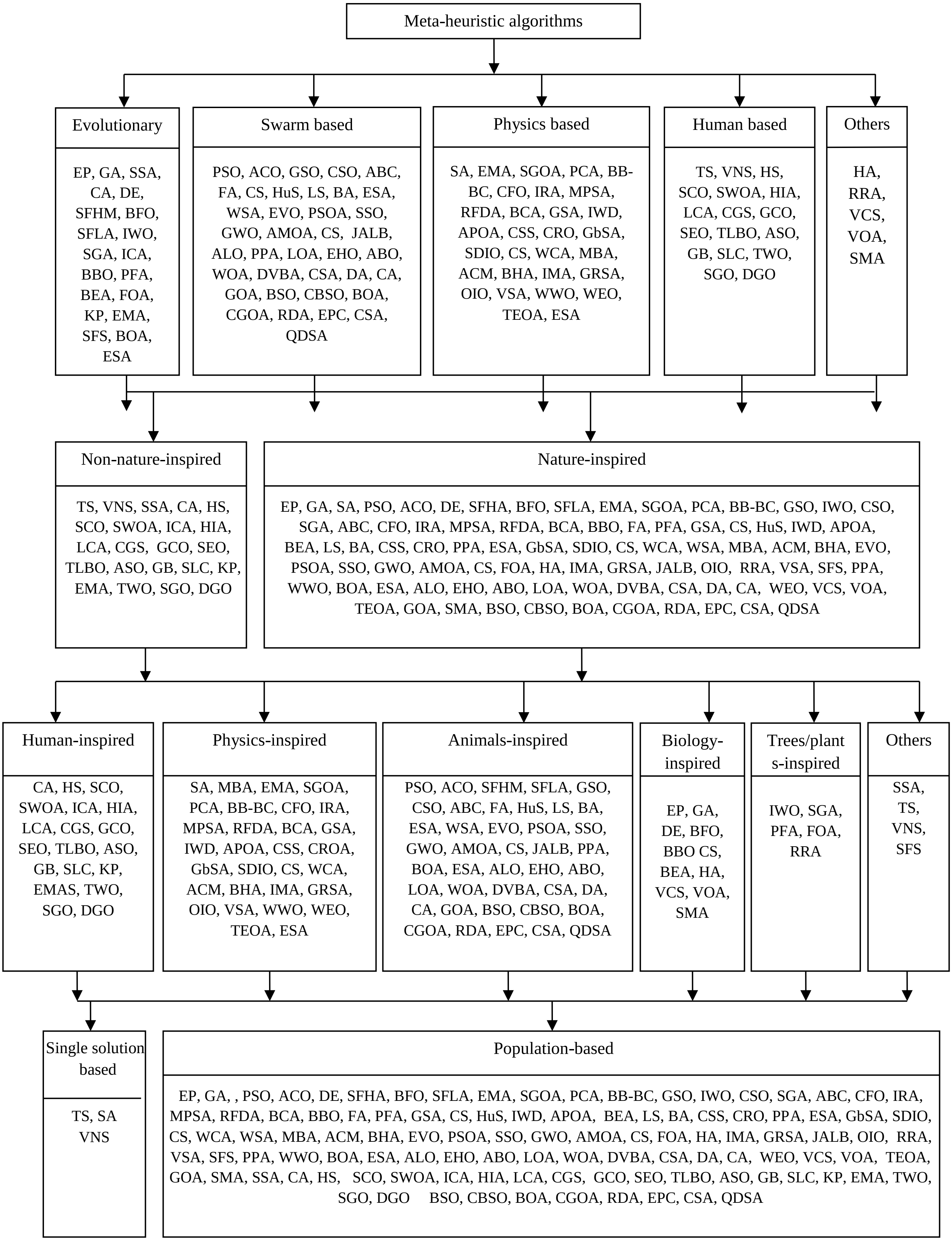

Fig. 4. *Classification of 110 investigated meta-heuristic algorithms.*

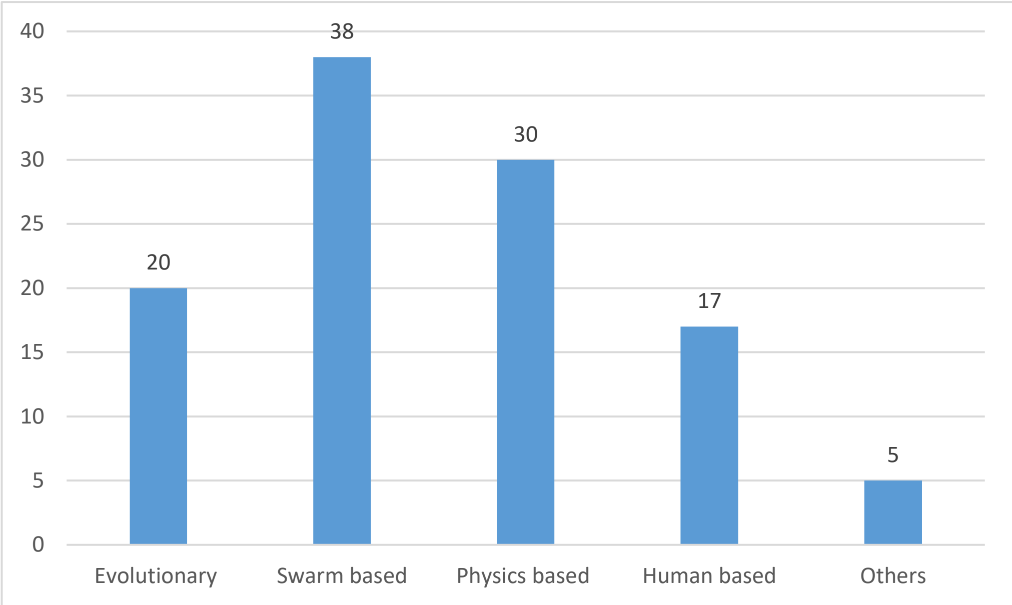

Fig. 5. *Classification of meta-heuristic algorithms based on their features statistically.*

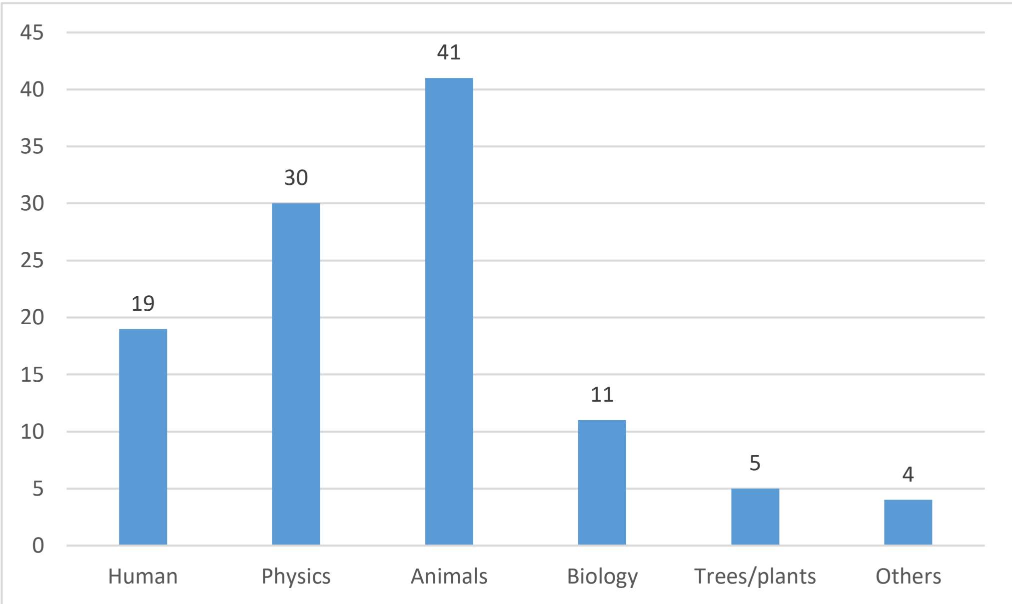

Fig. 6. *Classification of meta-heuristic algorithms based on the source of inspiration statistically.*

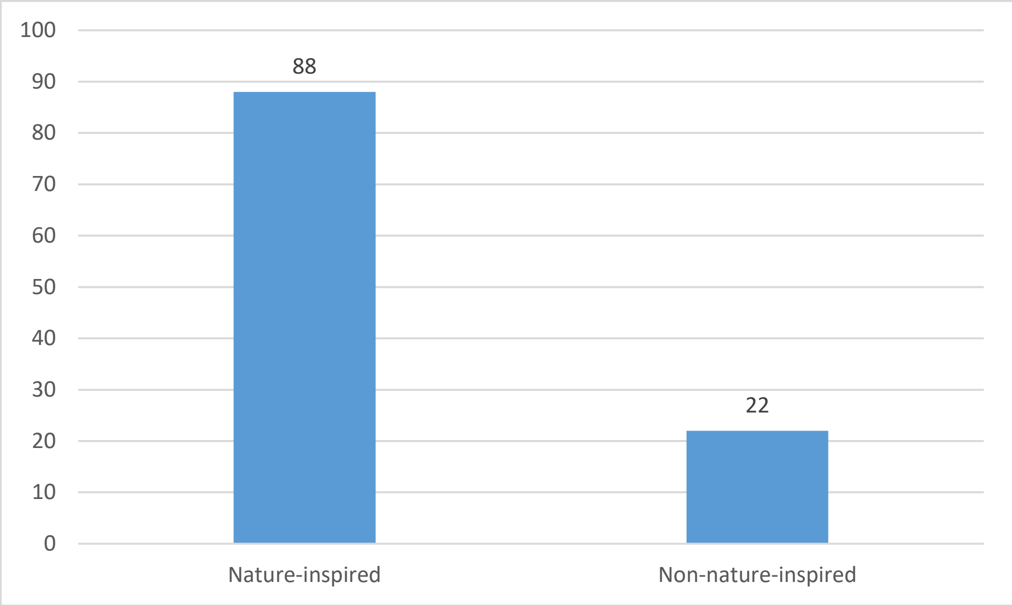

Fig. 7. *Classification of meta-heuristic algorithms based on nature/non-nature-inspired statistically.*

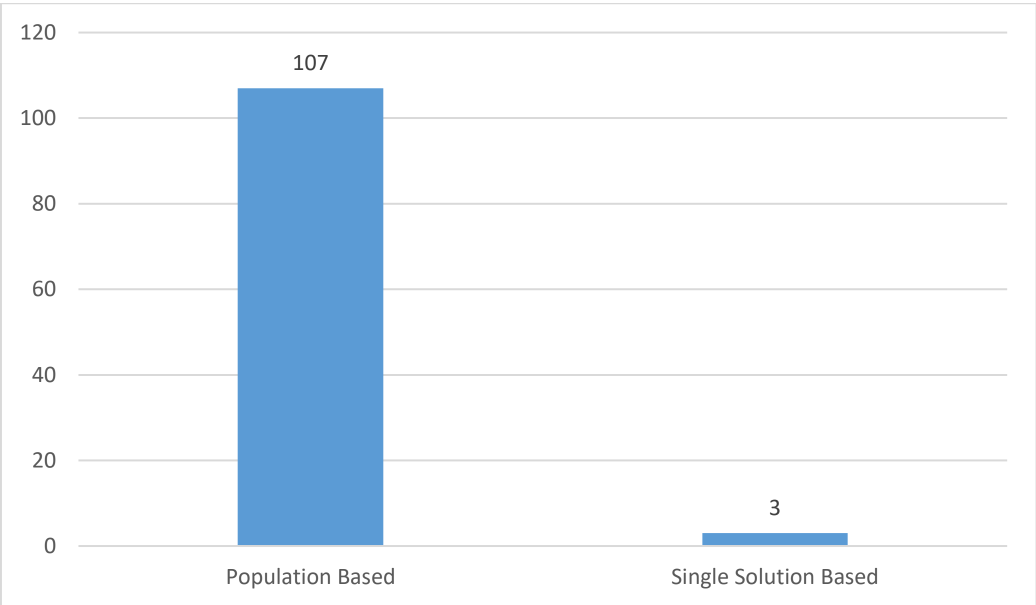

Fig. 8. *Classification of meta-heuristic algorithms based on population / single solution based statistically.*

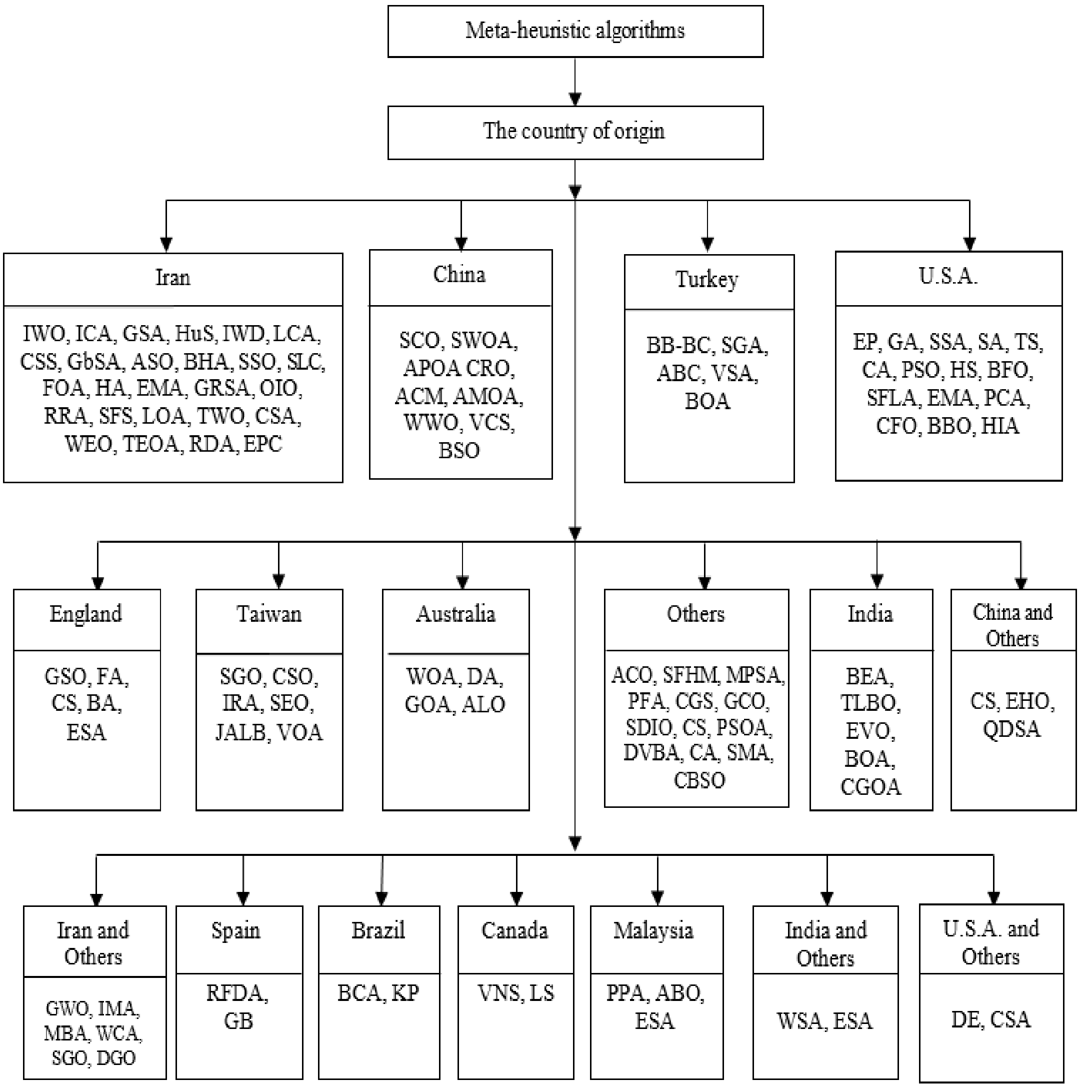

Fig. 9. *Classifications of 110 investigated meta-heuristic algorithms based on the country of origin.*

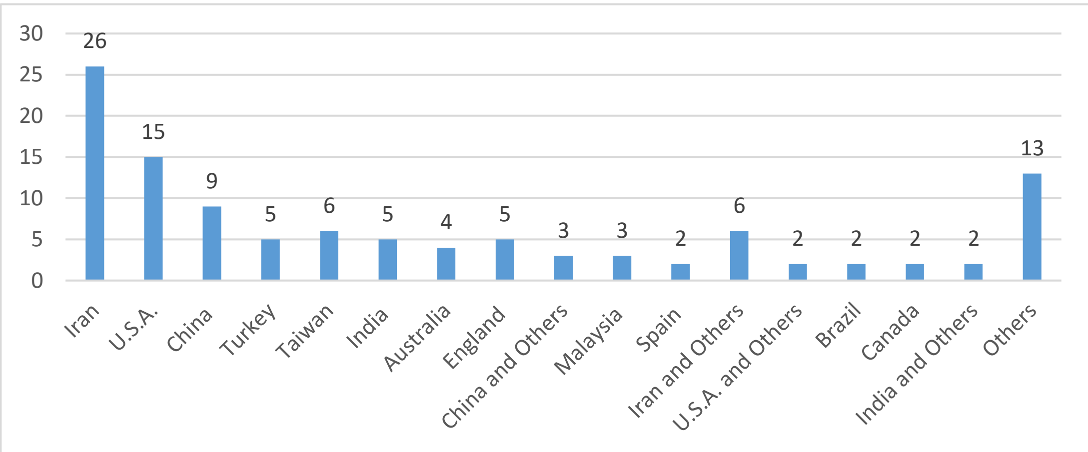

Fig. 10. *Classifications of (110 investigated) meta-heuristic algorithm based on the country of origin statistically.*

5. Conclusion

One of the main achievements of this paper is to categorize a large number of meta-heuristic algorithms (i.e. 110 algorithms) and give a brief explanation for each of them. Most of meta-heuristic algorithms have been introduced and developed over the last two decades (i.e. 91% of aforementioned algorithms have been introduced since 2000). Also, most of the meta-heuristic algorithms are inspired by nature in such a way that almost 80% of them are in this category. Moreover, 74% of algorithms are inspired by animal's behavior, physics, and biological laws. In addition, Iran is the country of origin for about 24% of algorithms and U.S.A is the country of origin for about 14% of algorithms. It should also be mentioned that, in this research, meta-heuristic algorithms are categorized from different aspects such as: nature-inspired vs. non-nature-inspired based, swarm based, evolutionary based, physics based, human based, source of inspiration, population based vs. single solution based and finally based on the country of origin. For further studies, it is suggested to compare and rank some meta-heuristic algorithms. It is also recommended to present a new meta-heuristic algorithm through being inspired by natural or unnatural phenomena.

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Numerical Approximation for the Fractional Advection-Diffusion Equation Using a High Order Difference Scheme

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ABSTRACT

In this paper, a one-dimensional fractional advection-diffusion equation is considered. First, we propose a numerical approximation of the Riemann-Liouville fractional derivative which is fourth-order accurate, then a numerical method for the fractional advection-diffusion equation using a high order finite difference scheme is presented. It is proved that the scheme is convergent. The stability analysis of numerical solutions is also discussed. The method is applied in several examples and the accuracy of the method is tested in terms of L_∞ error norm. Furthermore, the numerical results have been compared with some other methods.

1. Introduction

Fractional calculus is a useful mathematical tool for applied sciences. The fractional advection–diffusion equations provide an adequate and accurate description of the movement of solute in an aquifer. However, there are major obstacles that restrict their applications. From a modeling viewpoint, the fractional advection diffusion equation has been presented as a more suitable model for many problems that appear in different fields, such as engineering, physics, chemistry and hydrology.

Fractional differential equation are generalizations of classical differential equations of integer order that have recently proved to be valuable tools for the modelling of many physical phenomena and have been the focus of many studies due to their frequent appearances in various applications, such as physics, biology, finance and fractional dynamics, engineering, signal processing, and control theory [15].

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The fractional advection diffusion equation was given by

$$\frac{\partial u(x,t)}{\partial t} = -V\frac{\partial u(x,t)}{\partial x} + D\left(\frac{\partial^{\alpha} u(x,t)}{\partial x^{\alpha}} + \frac{\partial^{\alpha} u(x,t)}{\partial(-x)^{\alpha}}\right), \tag{1}$$

Where u is the concentration, V is the average velocity, x is the spatial coordinate, t is the time, D is the diffusion coefficient, α is the order of the fractional differentiation with $1<\alpha\le 2$. The fractional advection diffusion equation was later generalized by Benson et al. [16] to include the parameter β, given by

$$\frac{\partial u(x,t)}{\partial t} = -V\frac{\partial u(x,t)}{\partial x} + D\left(\frac{1}{2}+\frac{\beta}{2}\right)\frac{\partial^{\alpha} u(x,t)}{\partial x^{\alpha}} + D\left(\frac{1}{2}-\frac{\beta}{2}\right)\frac{\partial^{\alpha} u(x,t)}{\partial(-x)^{\alpha}}, \tag{2}$$

for $-1\le\beta\le 0$, the transition probability is skewed backward, while for $0\le\beta\le 1$ the transition probability is skewed forward. For $\beta=0$, we obtain the model presented in [17], which can be expressed as follows

$$\frac{\partial u(x,t)}{\partial t} = -V\frac{\partial u(x,t)}{\partial x}\nabla_{\alpha}^{\beta}u(x,t) + p(x,t) \tag{3}$$

where the fractional operator is given by

$$\nabla_{\alpha}^{\beta}u(x,t) = \left(\frac{1}{2}+\frac{\beta}{2}\right)\frac{\partial^{\alpha} u(x,t)}{\partial x^{\alpha}} + \left(\frac{1}{2}-\frac{\beta}{2}\right)\frac{\partial^{\alpha} u(x,t)}{\partial(-x)^{\alpha}}, \tag{4}$$

with initial condition

$u(x,0)=f(x), x\in\Re$

The Riemann–Liouville fractional derivatives of order α, for $x\in[a,b], -\infty\le a<b\le\infty$, are defined by

$$\frac{\partial^{\alpha} u}{\partial x^{\alpha}}(x,t) = \frac{1}{\Gamma(n-\alpha)}\frac{\partial^{n}}{\partial x^{n}}\int_{a}^{x} u(\xi,t)(x-\xi)^{n-\alpha-1}d\xi, n-1<\alpha<n. \tag{5}$$

$$\frac{\partial^{\alpha} u}{\partial(-x)^{\alpha}}(x,t) = \frac{(-1)^{n}}{\Gamma(n-\alpha)}\frac{\partial^{n}}{\partial x^{n}}\int_{x}^{b} u(\xi,t)(\xi-x)^{n-\alpha-1}d\xi, n-1<\alpha<n. \tag{6}$$

A meshless method of *Eq. (1)* has been discussed by Mardani et al. [1] and Tayebi et al. [2]. A functional variable method is given in Aminikhah et al. [4]. An first integral method for the solution of fractional differential equations is given in [5]. Lattice Boltzmann method [3]. And finite element methods for linear and nonlinear diffusion problems [7], [8], [9], [12], [13], and [14]. In this paper, by using a Lax–Wendroff-type time discretization procedure, we develop an explicit numerical method which is second order in time and space for fractional advection diffusion problems with source terms in unbounded and bounded domains with homogeneous boundary conditions. Since the numerical method is explicit, it is a more cost-effective method than the implicit schemes. Additionally, explicit methods are better tools for problems wherein advection plays an important role. The classical Lax–Wendroff method was derived for hyperbolic equations [18] and afterwards was extended for advection diffusion equations. This method uses a small stencil in time and also uses the original differential equation extensively, that is, the discretization procedure converts time derivatives in space derivatives. The layout of this paper is as follows. In Section 2, we will illustrate construction of the scheme and in Section 3 we describe the numerical method. In Section 4, we study the stability analysis. In the Section 5, we will prove the convergence of difference scheme analysis. The Section 6 includes some numerical tests which confirm the fourth-order convergence of the numerical method. A summary is given at the end of the paper in Section 7.

2. Construction of the Scheme

we present a new numerical approximation that this approximation is fourth-order accurate. Consider first the left derivative, that is,

$$\frac{\partial^{\alpha} u}{\partial x^{\alpha}}(x,t) = \frac{1}{\Gamma(2-\alpha)} \frac{\partial^2}{\partial x^2} \int_{-\infty}^{\infty} u(\xi,t)(x-\xi)^{1-\alpha} d\xi, 1<\alpha<2. \tag{7}$$

We define the mesh points $x_j = j\Delta x, j \in Z$ where $h = \Delta x$ denotes the uniform space step. For a fixed time t, let us denote

$$I_{\alpha}(x) = \int_{-\infty}^{x} u(\xi,t)(x-\xi)^{1-\alpha} d\xi. \tag{8}$$

First, we do the following approximation at x_j

$$\frac{\partial^2}{\partial x^2} I_{\alpha}(x_j) \approx \frac{-I_{\alpha}(x_{j+2}) + 16 I_{\alpha}(x_{j+1}) - 30 I_{\alpha}(x_j) + 16 I_{\alpha}(x_{j-1}) - I_{\alpha}(x_{j-2})}{12h^2}, \tag{9}$$

and $i_{\alpha}(x)$ defined by

$$i_{\alpha}(x_{j+1}) = \int_{-\infty}^{x_{j+1}} s_{j+1}(\xi)(x_{j+1}-\xi)^{1-\alpha} d\xi, \tag{10}$$

The spline $s_{j+1}(\xi)$ interpolates the points $\{(x_k,t) : k \le j+1\}$ and is of the form [15]

$$s_{j+1}(\xi) = \sum_{k=-\infty}^{j+1} u(x_k,t) s_{j+1,k}(\xi). \tag{11}$$

In each interval $[x_{k-2}, x_{k+2}]$ for $k \le j+1$ we have

$$s_{j+1,k}(\xi)=\begin{cases}\dfrac{\xi-x_{k-2}}{x_{k-1}-x_{k-2}} & x_{k-2}\le\xi\le x_{k-1}\\ \dfrac{\xi-x_{k-1}}{x_k-x_{k-1}} & x_{k-1}\le\xi\le x_k\\ \dfrac{\xi-x_k}{x_{k+1}-x_k} & x_k\le\xi\le x_{k+1}\\ \dfrac{x_{k+1}-\xi}{x_{k+2}-x_{k+1}} & x_{k+1}\le\xi\le x_{k+2}\\ 0 & \text{otherwise}\end{cases}. \tag{12}$$

And for $k=j+1$,

$$s_{j+1,j+1}=\begin{cases}\dfrac{\xi-x_{j+1}}{x_{j+2}-x_{j+1}} & x_j\le\xi\le x_{j+1}\\ 0 & \text{otherwise}\end{cases}. \tag{13}$$

From *Eqs. (10)* and *(11)*, we have

$$i_\alpha(x_{j+1})=\sum_{k=-\infty}^{j+1}u(x_k,t)\int_{x_{k-2}}^{x_{k+2}}s_{j+1,k}(\xi)(x_{j+1}-\xi)^{1-\alpha}d\xi. \tag{14}$$

Therefore,

$$\begin{aligned}\int_{x_{k-2}}^{x_{k+2}}s_{j+1,k}(\xi)(x_{j+1}-\xi)^{1-\alpha}d\xi=\int_{x_{k-2}}^{x_{k-1}}\frac{\xi-x_{k-2}}{h}(x_{j+1}-\xi)^{1-\alpha}d\xi+\int_{x_{k-1}}^{x_k}\frac{\xi-x_{k-1}}{h}(x_{j+1}-\xi)^{1-\alpha}d\xi+\\ \int_{x_k}^{x_{k+1}}\frac{\xi-x_k}{h}(x_{j+1}-\xi)^{1-\alpha}d\xi+\int_{x_{k+1}}^{x_{k+2}}\frac{x_{k+1}-\xi}{h}(x_{j+1}-\xi)^{1-\alpha}d\xi=\frac{\Delta x^{2-\alpha}}{(2-\alpha)(3-\alpha)}a_{j+1,k}.\end{aligned} \tag{15}$$

Where

$$a_{j+1,k}=\begin{cases}-(j-k-2)^{3-\alpha}+16(j-k-1)^{3-\alpha}-30(j-k)^{3-\alpha}+16(j-k+1)^{3-\alpha}-(j-k+2)^{3-\alpha} & k\le j\\ 1 & k=j+1\end{cases}. \tag{16}$$

Therefore,

$$i_\alpha(x_{j+1})=\frac{\Delta x^{2-\alpha}}{(2-\alpha)(3-\alpha)}\sum_{k=-\infty}^{j+1}u(x_k,t)a_{j+1,k}. \tag{17}$$

And an approximation for $\frac{\partial^2}{\partial x^2}I_\alpha(x_{j+1})$, is given by

$$\frac{-I_\alpha(x_{j+2})+16I_\alpha(x_{j+1})-30I_\alpha(x_j)+16I_\alpha(x_{j-1})-I_\alpha(x_{j-2})}{12h^2}.$$

That is

$$\frac{\Delta x^{-\alpha}}{12(2-\alpha)(3-\alpha)}\left[\begin{array}{l}\sum_{k=-\infty}^{j+2} u(x_k,t)a_{j+2,k}+16\sum_{k=-\infty}^{j+1} u(x_k,t)a_{j+1,k}-30\sum_{k=-\infty}^{j} u(x_k,t)a_{j,k}+\\ 16\sum_{k=-\infty}^{j-1} u(x_k,t)a_{j-1,k}-\sum_{k=-\infty}^{j-2} u(x_k,t)a_{j-2,k}\end{array}\right]. \tag{18}$$

We assume there are approximation U_j^n to the values $u(x_j,t_n)$, where $t_n = n\Delta t, n \geq 0$ and we define the fractional operator as

$$\delta_\alpha U_j^n = \frac{1}{12\Gamma(4-\alpha)}\sum_{k=-\infty}^{j+2} q_{j,k}U_k^n. \tag{19}$$

Where

$$\begin{aligned}
&q_{j,k} = -a_{j+2,k} + 16a_{j+1,k} - 30a_{j,k} + 16a_{j-1,k} - a_{j-2,k} \quad k \leq j-2\\
&q_{j,j-1} = -a_{j+2,j-1} + 16a_{j+1,j-1} - 30a_{j,j-1} + 16a_{j-1,j-1}\\
&q_{j,j} = -a_{j+2,j} + 16a_{j+1,j} - 30a_{j,j}\\
&q_{j,j+1} = -a_{j+2,j+1} + 16a_{j+1,j+1}\\
&q_{j,j+2} = -a_{j+2,j+2}.
\end{aligned} \tag{20}$$

Therefore, an approximation of *Eq. (7)* can be given by $\frac{\delta_\alpha U_j^n}{\Delta x^\alpha}$.We can also write the fractional operator *Eq. (19)* as

$$\delta_\alpha U_j^n = \frac{1}{12\Gamma(4-\alpha)}\sum_{m=-2}^{\infty} q_{j,j-m}U_{j-m}^n, \tag{21}$$

we define,

$$a_m = \begin{cases} -(m+2)^{3-\alpha} + 16(m+1)^{3-\alpha} - 30(m)^{3-\alpha} + 16(m-1)^{3-\alpha} - (m-2)^{3-\alpha} & m \geq 2 \\ 1 & m = 0 \end{cases}. \tag{22}$$

And

$$q_m = \begin{cases} -a_{m+2} + 16a_{m+1} - 30a_m + 16a_{m-1} - a_{m-2} & m \geq 2 \\ -a_3 + 16a_2 - 30a_1 + 16a_0 & m = 1 \\ -a_2 + 16a_1 - 30a_0 & m = 0 \\ -a_1 + 16a_0 & m = -1 \\ -a_0 & m = -2 \end{cases}, \tag{23}$$

we have

$$\delta_\alpha U_j^n = \frac{1}{12\Gamma(4-\alpha)}\sum_{m=-2}^{\infty} q_m U_{j-m}^n. \tag{24}$$

And similarly in the interval $[x,\infty)$

$$\delta'_\alpha U_j^n = \frac{1}{12\Gamma(4-\alpha)}\sum_{m=-2}^{\infty} q_m U_{j+m}^n. \tag{25}$$

3. Numerical Method

To derive a finite difference scheme, we suppose there are approximations $U^n = U_j^n$. to the values $u(x_j, t_n)$ at the mesh points $x_j = j\Delta x, h = \Delta x, j \in Z$ and $t_n = n\Delta t, n \geq 0$.

We assume $s = \frac{V\Delta t}{2\Delta x}$ and $\mu = \frac{\Delta t D}{\Delta x^\alpha}$,

we expand u about time level n, that is, $t = n\Delta t$ to obtain

$$u(x,t_{n+1}) - u(x,t_n) = \Delta t \frac{\partial u}{\partial t}(x,t_n) + \frac{\Delta t^2}{2}\frac{\partial^2 u}{\partial t^2}(x,t_n) + O(\Delta t^3). \tag{26}$$

Then, from *Eq. (26)* we have

$$\frac{\partial^2 u}{\partial t^2}(x,t) = -V(x,t)\frac{\partial^2 u}{\partial x \partial t}(x,t) + d(x,t)\nabla_\alpha^\beta\left(\frac{\partial u}{\partial t}(x,t)\right) + p_t(x,t). \tag{27}$$

And

$$\frac{\partial^2 u}{\partial t^2}(x,t) \approx V^2(x,t)\frac{\partial^2 u}{\partial x^2}(x,t) - V(x,t)p_x(x,t) + p_t(x,t). \tag{28}$$

Inserting *Eqs.(3)* and *(29)* into *Eq.(26)* gives

$$\begin{aligned} u(x,t_{n+1}) \approx{}& u(x,t_n) + \Delta t\left(-V(x,t)\frac{\partial u}{\partial x}(x,t_n) + d(x,t)\nabla_\beta^\alpha u(x,t_n) + p(x,t_n)\right) + \\ & \frac{\Delta t^2}{2}\left(V^2(x,t)\frac{\partial^2 u}{\partial x^2}(x,t_n) - V(x,t)p_x(x,t) + p_t(x,t)\right). \end{aligned} \tag{29}$$

Therefore,

$$\begin{aligned} u(x,t_{n+1}) \approx{}& u(x,t_n) - V(x,t)\Delta t\frac{\partial u}{\partial x}(x,t_n) + \Delta t d(x,t)\nabla_\beta^\alpha u(x,t_n) + \\ & \frac{\Delta t^2}{2}V^2(x,t)\frac{\partial^2 u}{\partial x^2}(x,t_n) + \Delta t\left(p\ (x,t) + \frac{\Delta t}{2}\left(-V(x,t)p_x(x,t) + p_t(x,t)\right)\right). \end{aligned} \tag{30}$$

We define the following operators,

$$\frac{\partial u}{\partial x}(x,t_n)=\frac{-U_{j+2}^n+8U_{j+1}^n-8U_{j-1}^n+U_{j-2}^n}{12h}+o(\Delta x^4),$$
$$\frac{\partial^2 u}{\partial x^2}(x,t_n)=\frac{-U_{j+2}^n+16U_{j+1}^n-30U_j^n+16U_{j-1}^n-U_{j-2}^n}{12h^2}+o(\Delta x^4). \tag{31}$$

And the fractional operator

$$\delta_\beta^\alpha u(x,t_n)=\left(\frac{1}{2}+\frac{\beta}{2}\right)\delta_\alpha u(x,t_n)+\left(\frac{1}{2}-\frac{\beta}{2}\right)\delta'_\alpha u(x,t_n). \tag{32}$$

We introduce the following notations

$$\delta U_j^n=-U_{j+2}^n+8U_{j+1}^n-8U_{j-1}^n+U_{j-2}^n$$
$$\delta^2 U_j^n=-U_{j+2}^n+16U_{j+1}^n-30U_j^n+16U_{j-1}^n-U_{j-2}^n. \tag{33}$$

We obtain

$$\overline{p}_j^n=p_j^n+\frac{\Delta t}{2}\left(-V(x,t)\frac{-U_{j+2}^n+8U_{j+1}^n-8U_{j-1}^n+U_{j-2}^n}{12h}+\frac{U_j^{n+1}-U_j^n}{\Delta t}\right).$$

Where

$$p_j^n=p(x_j,t_n). \tag{34}$$

Therefore, from *Eq. (30)* we have

$$U_j^{n+1}=U_j^n-\frac{s}{6}\delta U_j^n+\mu\delta_\beta^\alpha U_j^n+\frac{1}{6}s^2\delta^2 U_j^n+\Delta t\overline{p}_j^n. \tag{35}$$

4. Stability

If u_j^n is the exact solution $u(x_j,t_n)$, let U_j^n be a perturbation of u_j^n. The perturbation error

$$e_j^n=U_j^n-u_j^n,$$

will be propagated forward in time according to the equation

$$e_j^{n+1}=e_j^n-\frac{s}{6}\delta e_j^n+\mu\delta_\beta^\alpha e_j^n+\frac{1}{6}s^2\delta^2 e_j^n. \tag{36}$$

Theorem 1. For $\beta=0$, the numerical method is von Neumann stable if, and only if,

$$\frac{16}{3}s^2+\frac{\mu}{12\Gamma(4-\alpha)}(\cos\theta+1)(q_{-2}+q_2)+\frac{\mu}{24\Gamma(4-\alpha)}(q_{-1}+q_1)\le-\frac{1}{12}. \tag{37}$$

Proof. We have

$$G(\theta)=1+\frac{s}{6}i(-4\sin^2\theta+16\sin\theta)+\frac{s^2}{6}(-4\cos^2\theta+32\cos\theta-30)+$$
$$\frac{\mu}{24\Gamma(4-\alpha)}\sum_{m=-2}^{\infty}q_m\cos(m\theta).$$

That is

$$|G(\theta)|^2=\left[1+\frac{s^2}{6}(-4\cos^2\theta+32\cos\theta-30)+\frac{\mu}{24\Gamma(4-\alpha)}\sum_{m=-2}^{\infty}q_m\cos(m\theta)\right]^2$$
$$-\left[\frac{s}{6}(-4\sin^2\theta+16\sin\theta)\right],^2$$

if we have *Condition (37)* then $|G(\theta)|\le 1$, for all θ. We have that

$$\sum_{m=-2}^{\infty}q_m\cos(m\theta)\le(q_{-2}+q_2)\cos 2\theta+(q_{-1}+q_1)\cos\theta+q_0+\sum_{m=3}^{\infty}q_m=$$
$$2(q_{-2}+q_2)(\cos^2\theta-1)+(q_{-1}+q_1)(\cos\theta-1).$$

Therefore,

$$|G(\theta)|^2\le\left[\begin{array}{l}1-\frac{s^2}{3}(2\cos^2\theta-16\cos\theta+15)+\frac{\mu}{12\Gamma(4-\alpha)}(\cos^2\theta-1)(q_{-2}+q_2)+\\ \frac{\mu}{24\Gamma(4-\alpha)}(\cos\theta-1)(q_{-1}+q_1)\end{array}\right]^2$$
$$-\left[\frac{s}{6}(-4\sin^2\theta+16\sin\theta)\right]^2.$$

Then,

$$|G(\theta)|^2\le$$
$$\left[1-\frac{s^2}{3}(2\cos^2\theta-1)+(\cos\theta-1)\left(\frac{16}{3}s^2+\frac{\mu}{12\Gamma(4-\alpha)}(\cos\theta+1)(q_{-2}+q_2)+\frac{\mu}{24\Gamma(4-\alpha)}(q_{-1}+q_1)\right)\right]^2$$
$$-\left[\frac{s}{6}(-4\sin^2\theta+16\sin\theta)\right].^2$$

For $K=\left(\frac{16}{3}s^2+\frac{\mu}{12\Gamma(4-\alpha)}(\cos\theta+1)(q_{-2}+q_2)+\frac{\mu}{24\Gamma(4-\alpha)}(q_{-1}+q_1)\right)$.

It follows

$$|G(\theta)|^2 \le \left[1-\frac{s^2}{3}(2\cos^2\theta-1)+(\cos\theta-1)K\right]^2-\left[\frac{s}{6}(-4\sin^2\theta+16\sin\theta)\right]^2$$
$$\le \left[1-\frac{s^2}{3}(2\cos^2\theta-1)+(\cos\theta-1)K\right]^2=1+\frac{s^4}{9}(2\cos^2\theta-1)-\frac{2Ks^2}{3}(\cos\theta-1)(2\cos^2\theta-1).$$

Note that

$$\left[\sum_{m=3}^{\infty}(-1)^m q_m+q_0\right]\le 0.$$

Then

$$K=\left(\frac{16}{3}s^2+\frac{\mu}{12\Gamma(4-\alpha)}(\cos\theta+1)(q_{-2}+q_2)+\frac{\mu}{24\Gamma(4-\alpha)}(q_{-1}+q_1)\right)\le$$
$$\frac{16}{3}+\frac{\mu}{6\Gamma(4-\alpha)}\left[\sum_{m=3}^{\infty}(-1)^m q_m+q_0\right]\le\frac{16}{3}.$$

And

$$|G(\theta)|^2\le 1+\frac{s^4}{9}(2\cos^2\theta-1)-\frac{2Ks^2}{3}(\cos\theta-1)(2\cos^2\theta-1).$$

We know that $|G(\theta)|\le 1$ is stability condition; therefore, it is necessary that

$$\frac{2Ks^2}{3}(\cos\theta-1)(2\cos^2\theta-1)\ge\frac{s^4}{9}(2\cos^2\theta-1).$$

Therefore, for $K\le-\frac{1}{12}$, we have $|G(\theta)|^2\le 1$, for all θ .

5. Convergence Analysis

In this section we analyze the convergence of the numerical method using the framework of consistency and stability. We have the global error given by $e^n=u^n-U^n$, where u^n and U^n are respectively exact and approximate solutions. The truncation error at each discrete point x_j , is given by

$$E_j^n=\frac{u_j^{n+1}-u_j^n}{\Delta t}+\frac{V}{6}\frac{-U_{j+2}^n+8U_{j+1}^n-8U_{j-1}^n+U_{j-2}^n}{\Delta t}-$$
$$\frac{V^2\Delta t}{6}\frac{-U_{j+2}^n+16U_{j+1}^n-30U_j^n+16U_{j-1}^n-U_{j-2}^n}{h^2}-\frac{D}{2\Delta x^\alpha}\delta_\beta^\alpha u_j^n=$$
$$\left(\frac{\partial u}{\partial t}\right)_j^n+\frac{\Delta t}{2}\left(\frac{\partial^2 u}{\partial t^2}\right)_j^n+O(\Delta t^2)+V\left(\frac{\partial u}{\partial x}\right)_j^n+O(\Delta x^4)-$$
$$\frac{V^2\Delta t}{6}\left(\frac{\partial^2 u}{\partial x^2}\right)_j^n+O(\Delta x^4)-D(\nabla_\beta^\alpha u)_j^n+O(\Delta x^4).$$

Therefore,

$$E_j^n=\left(\frac{\partial u}{\partial t}\right)_j^n+O(\Delta t^2)+V\left(\frac{\partial u}{\partial x}\right)_j^n+O(\Delta x^4)-\frac{V^2\Delta t}{6}\left(\frac{\partial^2 u}{\partial x^2}\right)_j^n+O(\Delta x^4)-D(\nabla_\beta^\alpha u)_j^n+O(\Delta x^4).$$

6. Numerical Experiments

In this section we consider examples that are solved by our presented method in previous sections. In order to illustrate the accuracy of the method, we used the error norm L_∞ which is defined as follows

$$L_\infty=\max_j\left|U(x_j,t)-u(x_j,t)\right|.$$

And we compare our results with the results in [10].

Example 1. We assume $\beta=1$ in *Eq. (3)*, that is, we have the equation

$$\frac{\partial u(x,t)}{\partial t}=-V\frac{\partial u(x,t)}{\partial x}+D\nabla_\alpha^\beta u(x,t)+p(x,t). \tag{38}$$

In the domain $0\le x\le 1$, we assume the problem has initial condition $u(x,0)=x^4$ and boundary conditions $u(0,t)=0, u(1,t)=e^{-t}$. Let $V=0.2$ $D=\frac{\Gamma(5-\alpha)}{24}$ And $p(x,t)=e^{-t}x^3(4V-x-x^{1-\alpha})$.

The exact solution is given by $u(x,t)=e^{-t}x^4$.

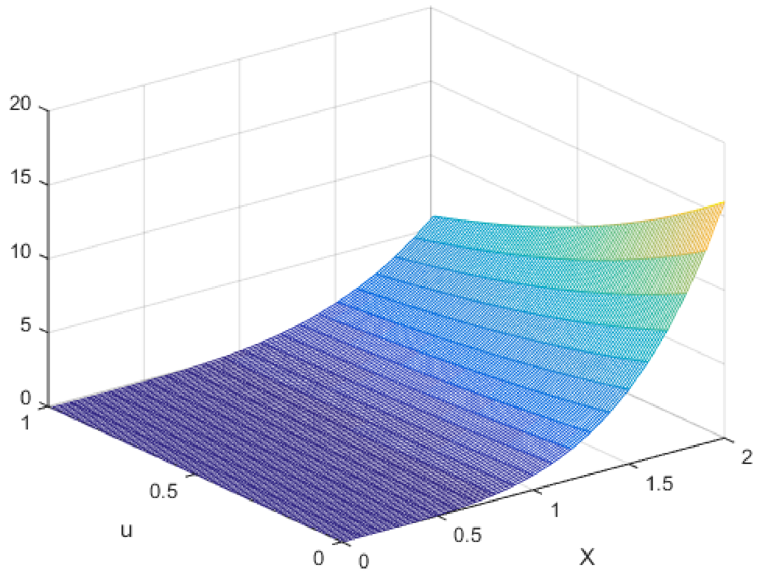

Fig. 1. *Shows the exact solution of Eq. (38) when* $\beta=1$.

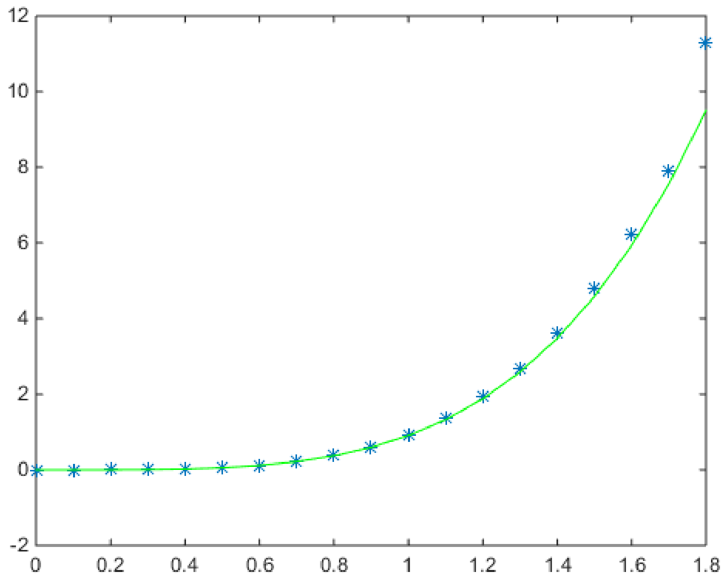

Fig. 2. *Comparison of the exact and numerical solution with* $\alpha = 1.2, \beta = 1$.

Table 1. *The computational results for example 1 by our method and method in [10] and [3].*

		$\alpha = 1.2$	$\alpha = 1.4$	$\alpha = 1.6$	$\alpha = 2$
Our method $\Delta x = 0.1$	L_∞	0.2376×10^{-7}	0.17609×10^{-7}	0.2151×10^{-7}	0.16705×10^{-7}
$\Delta x = 0.01$	L_∞	0.2256×10^{-9}	0.1741×10^{-9}	0.2004×10^{-9}	0.1079×10^{-9}
Method in [10] $\Delta x = 0.1$	L_∞	0.4603×10^{-3}	0.3208×10^{-3}	0.4453×10^{-3}	0.1239×10^{-2}
$\Delta x = 0.01$	L_∞	0.5667×10^{-4}	0.4444×10^{-4}	0.4864×10^{-4}	0.1095×10^{-4}
Method in [3] $\Delta x = 0.1$	L_∞	0.5512×10^{-5}	0.3209×10^{-5}	0.1729×10^{-6} 0.7412×10^{-5}	0.1782×10^{-5}
$\Delta x = 0.01$	L_∞	0.1920×10^{-7}	0.6724×10^{-7}	0.4132×10^{-7}	

Table 1 shows the errors of our proposed method with values of h at different final times using L_∞ . Numerical results of this table confirm that the method has fourth-order of accuracy in temporal and spatial components, respectively. Comparison of this method to other methods, [10] and [3], confirms the efficiency and high accuracy of our proposed method.

The second example considers *Eq. (3)* for $\beta = 0$, that is, we have the equation

$$\frac{\partial u(x,t)}{\partial t} = -V\frac{\partial u(x,t)}{\partial x} + \frac{D}{2}\left(\frac{\partial^\alpha u(x,t)}{\partial x^\alpha} + \frac{\partial^\alpha u(x,t)}{\partial(-x)^\alpha}\right) + p(x,t). \quad (39)$$

<u>*Example 2.*</u> The second example considers *Eq. (3)* for $\beta = 0$, that is, we have the equation

$$\frac{\partial u(x,t)}{\partial t}=-V\frac{\partial u(x,t)}{\partial x}+\frac{D}{2}\left(\frac{\partial^{\alpha}u(x,t)}{\partial x^{\alpha}}+\frac{\partial^{\alpha}u(x,t)}{\partial(-x)^{\alpha}}\right)+p(x,t).$$

In the domain $0\le x\le 2$. We assume the initial condition is $u(x,0)=4x^2(2-x)^2$ and the boundary conditions are $u(0,t)=0, u(2,t)=0$.

Let $V=0.05$ $D=\frac{\Gamma(5-\alpha)}{2}$ and

$$p(x,t)=4e^{-t}(-x^2(2-x)^2+4Vx(x^2-3x+2)-x^{2-\alpha}A(x,\alpha)-(2-x)^{2-\alpha}B(x,\alpha)).$$

Where

$$A(x,\alpha)=2\alpha(\alpha-1)-6\alpha(2-x)+6(2-x)^2$$
$$B(x,\alpha)=2\alpha(\alpha-1)-6\alpha x+6x^2.$$

The exact solution is given by $u(x,t)=4e^{-t}x^2(2-x)^2$.

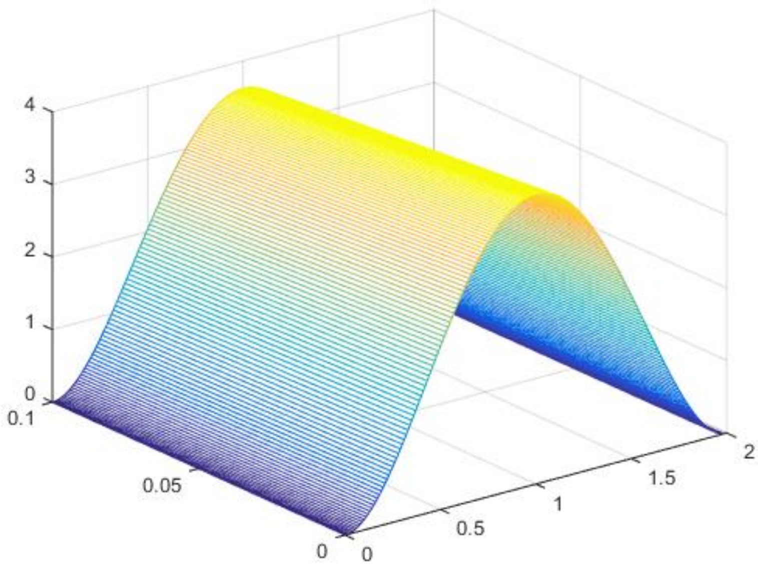

Fig. 3. *Shows the numerical approximation of Eq. (39) when $\beta=0$.*

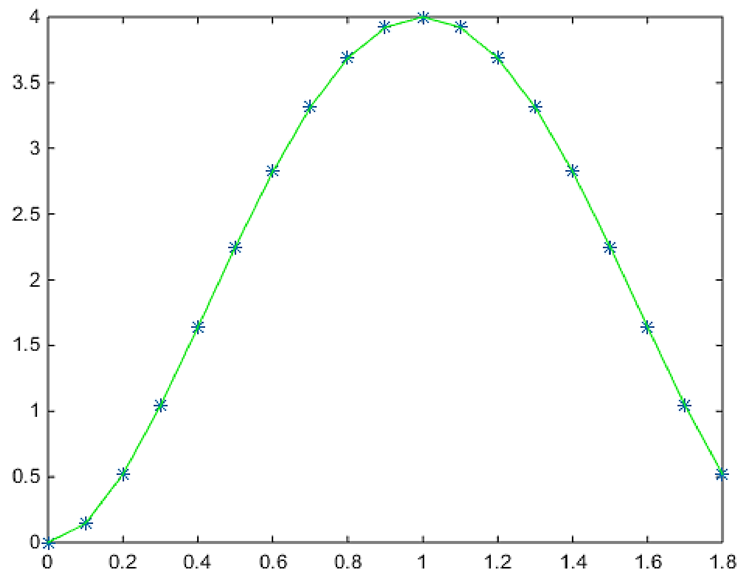

Fig. 4. *Comparison of the exact and numerical solution with* $\beta = 0, \Delta x = 0.1$.

Table 2. *The computational results for example 2.*

			$\alpha = 1.2$	$\alpha = 1.4$	$\alpha = 1.6$	$\alpha = 2$
Our method	$\Delta x = 0.1$	L_∞	0.7823×10^{-5}	0.1240×10^{-5}	0.1782×10^{-5}	0.4512×10^{-5}
	$\Delta x = 0.01$	L_∞	0.2361×10^{-8}	0.1421×10^{-8}	0.2344×10^{-8}	0.1761×10^{-8}
Method [6]	$\Delta x = 0.1$	L_∞	0.3216×10^{-3}	0.14213×10^{-3}	0.1214×10^{-3}	0.2325×10^{-3}
	$\Delta x = 0.01$	L_∞	0.7451×10^{-5}	0.4617×10^{-5}	0.1238×10^{-5}	0.2113×10^{-5}

According to *Table 2* the results indicate that our supposed scheme has a high accuracy and shows that errors are very small. our method is conditionally stable, consistent and convergent, which is fourth-order accurate with respect to the space step and second – order accurate to the time step.

Therefore, our method is more convenient than the [6], which is second order accurate with respect to the space step.

7. Conclusion

In this work, we have applied a high order finite difference method for the fractional advection-diffusion equation. Furthermore, we proved that this scheme is stable and convergent. Numerical results of the above tables confirm that the method has fourth-order of accuracy in temporal and spatial components, and the errors are very small. This scheme is an accurate and efficient approach for the solution of such types of nonlinear partial differential equations, we suggest to use this method for solving nonlinear equations [6] and [11]. We also see that the method presented produces very good results compared with the second order and fourth order methods proposed in [10] and [3]. Comparison of this method to other methods confirms the efficiency and high accuracy of our proposed method.

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