

A Generalized Model for Fuzzy Linear Programs with Trapezoidal Fuzzy Numbers

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PAPER INFO	ABSTRACT
<p>Chronicle: Received: 06 June 2017 Revised: 10 August 2017 Accepted: 24 August 2017 Available: 24 August 2017</p> <p>Keywords : Fuzzy linear programming. Fuzzy arithmetic. Fuzzy ordering. Fuzzy primal simplex algorithm.</p>	<p>In this paper, a linear programming problem with symmetric trapezoidal fuzzy number which is introduced by Ganesan et al. in [4] is generalized to a general kind of trapezoidal fuzzy number. In doing so, we first establish a new arithmetic operation for multiplication of two trapezoidal fuzzy numbers. In order to prepare a method for solving the fuzzy linear programming and the primal simplex algorithm, a general linear ranking function has been used as a convenient approach in the literature. In fact, our main contribution in this work is based on 3 items: 1) Extending the current fuzzy linear program to a general kind which doesn't essentially include the symmetric trapezoidal fuzzy numbers, 2) Defining a new multiplication role of two trapezoidal fuzzy numbers, 3) Establishing a fuzzy primal simplex algorithm for solving the generalized model. We in particular emphasize that this study can be used for establishing fuzzy dual simplex algorithm, fuzzy primal-dual simplex algorithm, fuzzy multi objective linear programming and the other similar methods which are appeared in the literature.</p>

1. Introduction

Fuzzy mathematical programming has been developed for treating uncertainty about the setting optimization problems. In recent years, various attempts have been made to study the solution of fuzzy linear programming problems, either from theoretical or computational point of view. After the pioneering works on this area many authors have considered various kinds of the FLP problems and have proposed several approaches to solve these problems [2,3,8,9,10,13]. Some authors have made a comparison between fuzzy numbers and in particular linear ranking function to solve the fuzzy linear programming problems. Of course, ranking functions have been proposed by researchers to meet their requirements regarding the

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problem under consideration and conceivably there are no generally accepted criteria for application of ranking functions. Using the concept of comparison of fuzzy numbers, Maleki et al. [10] proposed a new method to solve Fuzzy Number Linear Programming (FNLP) problems. Then, Mahdavi-Amiri and Nasserri [9] used a certain linear ranking function to define the dual of FNLP problems as a concept that gives an efficient method called the dual simplex algorithm [11] for solving FNLP problems. Also, Mahdavi-Amiri and Nasserri [8] proposed another approach to define dual of FNLP problems as Linear Programming with Fuzzy Variables (FVLP) problems. It introduced a dual simplex algorithm for solving FVLP problems. Nasserri and Ebrahimnejad [12] suggested a fuzzy primal simplex algorithm in order to solve the flexible linear programming problem. Next they recommended the fuzzy primal simplex method to solve the flexible linear programming problems directly without solving any auxiliary problem. Hosseinzadeh Lofi et al. [5] discussed full fuzzy linear programming (FFLP) problems whose parameters and variables are triangular fuzzy numbers. They used the concept of the symmetric triangular fuzzy number and introduced an approach to defuzzify a general fuzzy quantity. In order to deal with the problem, first of all, the fuzzy triangular number is approximated to its nearest symmetric triangular number, on the assumption that all decision variables are symmetric triangular. Kumar et al. [7] proposed a new method to find the fuzzy optimal solution of the same type of fuzzy linear programming problems. Ganesan and Veeramani [4] introduced a new type of fuzzy arithmetic for symmetric trapezoidal fuzzy numbers and proposed a method for solving FLP problems without converting them to the crisp linear programming problems. After that, Sapan Kumar Das and et al. [1] proposed a mathematical model for solving fully fuzzy linear programming problem with trapezoidal fuzzy numbers. However, the proposed approach could modify the mentioned model to a simpler one while it couldn't be applied for the general kind of the trapezoidal fuzzy numbers. Hence, in this paper, we first extended the recently quoted model by Ganesan and Veeramani to the general model, and then based it on a new multiplication role of two trapezoidal fuzzy numbers. We then established a fuzzy primal simplex algorithm to solve the mentioned generalized model. We also emphasized that the proposed approach could be used for re-establishing the similar format of solving algorithm such as fuzzy two-phase method, fuzzy dual simplex algorithm etc. We finally illustrated our work by dealing with some numerical examples which are convenient samples of problems. It will be observed that this is a more appropriate method for the decision

makers to adapt to the real situations and it, in particular, can solve these problems as simple as possible.

2. Preliminaries

2.1. Definitions and notations

In this section, some notations, concepts, new definitions and some fundamental results have been presented on fuzzy arithmetic in the lake of symmetric assumption and a new generalized definition has been particularly proposed for multiplication of the general trapezoidal fuzzy numbers.

Definition 2.1. A fuzzy set \tilde{a} on \mathbb{R} is said to be a trapezoidal fuzzy number if there exists real numbers a_1, a_2 , where $a_1 \leq a_2$ and $h_1, h_2 \geq 0$ such that

$$\tilde{a}(x) = \begin{cases} \frac{x}{h_1} + \frac{h_1 - a_1}{h_1}, & \text{for } x \in (a_1 - h_1, a_1) \\ 1, & \text{for } x \in (a_1, a_2) \\ \frac{-x}{h_2} + \frac{a_2 + h_2}{h_2}, & \text{for } x \in (a_2, a_2 + h_2) \\ 0, & \text{otherwise,} \end{cases}$$

where $\tilde{a}(x)$ is the membership function of fuzzy number \tilde{a} .

We denoted it by $\tilde{a} = [a_1, a_2, h_1, h_2]$.

In the above definition, when $h_1 = h_2$ we called it as "symmetric trapezoidal fuzzy number".

Moreover, when $h_1 = h_2 = 0$; $\tilde{a} = [a_1, a_2]$.

We use $\mathcal{F}(\mathbb{R})$ to denote the set of all trapezoidal fuzzy numbers.

Definition 2.2. Let $\tilde{a} = [a_1, a_2, h_1, h_2]$ and $\tilde{b} = [b_1, b_2, k_1, k_2]$ be two trapezoidal fuzzy numbers. If so, the arithmetic operations on \tilde{a} and \tilde{b} will be defined by:

- (i) Addition: $\tilde{a} + \tilde{b} = [a_1, a_2, h_1, h_2] + [b_1, b_2, k_1, k_2] = [a_1 + b_1, a_2 + b_2, h_1 + k_1, h_2 + k_2]$.
- (ii) Subtraction: $\tilde{a} - \tilde{b} = [a_1, a_2, h_1, h_2] - [b_1, b_2, k_1, k_2] = [a_1 - b_2, a_2 - b_1, h_1 + k_2, h_2 + k_1]$.

2.2. Ranking functions

One convenient approach for solving the FLP problems is based on the concept of comparison of fuzzy numbers by use of ranking functions (see [14]). An effective approach

for ordering the elements of $\mathcal{F}(\mathbb{R})$ is to define a ranking function $\mathcal{R}: \mathcal{F}(\mathbb{R}) \rightarrow \mathbb{R}$ which maps each fuzzy number into the real line, where a natural order exists.

Definition 2.3. We defined orders on $\mathcal{F}(\mathbb{R})$ by

$$\tilde{a} \succcurlyeq \tilde{b} \text{ if and only if } \mathcal{R}(\tilde{a}) \geq \mathcal{R}(\tilde{b}),$$

$$\tilde{a} \succ \tilde{b} \text{ if and only if } \mathcal{R}(\tilde{a}) > \mathcal{R}(\tilde{b}),$$

$$\tilde{a} \simeq \tilde{b} \text{ if and only if } \mathcal{R}(\tilde{a}) \simeq \mathcal{R}(\tilde{b}),$$

where \tilde{a} and \tilde{b} are in $\mathcal{F}(\mathbb{R})$. Also we write $\tilde{a} \preccurlyeq \tilde{b}$ if and only if $\tilde{b} \succcurlyeq \tilde{a}$.

Attention has been exclusively paid to linear ranking functions, that is, a ranking function \mathcal{R} such that:

$$\mathcal{R}(k\tilde{a} + \tilde{b}) = k\mathcal{R}(\tilde{a}) + \mathcal{R}(\tilde{b})$$

for any \tilde{a} and \tilde{b} belonging to $\mathcal{F}(\mathbb{R})$ and any $k \in \mathbb{R}$.

Now using the above approach, we may rank a big category of fuzzy numbers, where the symmetric property is extended to the non-symmetric form, too.

Remark 1. For any trapezoidal fuzzy number \tilde{a} , the relation $\tilde{a} \succcurlyeq \tilde{0}$ holds, if there exists $\varepsilon > 0$ and $\alpha > 0$ such that $\tilde{a} \succcurlyeq (-\varepsilon, \varepsilon, \alpha, \alpha)$. We realized that $\mathcal{R}(-\varepsilon, \varepsilon, \alpha, \alpha) = 0$. We also considered $\tilde{a} \approx \tilde{0}$ only if $\mathcal{R}(\tilde{a}) = 0$. Thus, without loss of generality, throughout the paper, we regard $\tilde{0} = (0, 0, 0, 0)$ as the zero trapezoidal fuzzy number.

The following lemma is now immediately at hand.

Lemma 2.1. Let \mathcal{R} be any linear ranking function. Then,

- (i) $\tilde{a} \succcurlyeq \tilde{b}$ if and only if $\tilde{a} - \tilde{b} \succcurlyeq \tilde{0}$ if and only if $-\tilde{b} \succcurlyeq -\tilde{a}$,
- (ii) If $\tilde{a} \succcurlyeq \tilde{b}$ and $\tilde{c} \succcurlyeq \tilde{d}$ then $\tilde{a} + \tilde{c} \succcurlyeq \tilde{b} + \tilde{d}$.

The linear ranking function has been considered on $\mathcal{F}(\mathbb{R})$ as

$$\mathcal{R}(\tilde{a}) = c_l a^l + c_u a^u + c_\alpha \alpha + c_\beta \beta,$$

where $\tilde{a} = (a^l, a^u, \alpha, \beta)$, and $c_l, c_u, c_\alpha, c_\beta$ are constants, at least one of them is nonzero. A special version of the above linear ranking function was first proposed by Yager [14] (see also [8] and [9]).

Proposition 2.1. For any trapezoidal fuzzy numbers \tilde{a}, \tilde{b} and \tilde{c} , we have

$$(i) \quad \tilde{c} (\tilde{a} + \tilde{b}) \approx (\tilde{c} \tilde{a} + \tilde{c} \tilde{b}),$$

$$(ii) \quad \tilde{c} (\tilde{a} - \tilde{b}) \approx (\tilde{c} \tilde{a} - \tilde{c} \tilde{b}).$$

Theorem 2.1. (i) The relation \preceq is a partial order relation on the set of trapezoidal fuzzy numbers.

(ii) The relation \preceq is a linear order relation on the set of trapezoidal fuzzy numbers.

(iii) For any two trapezoidal fuzzy numbers \tilde{a} and \tilde{b} ; if $\tilde{a} \preceq \tilde{b}$, then $\tilde{a} \preceq (1 - \lambda)\tilde{a} + \lambda\tilde{b} \preceq \tilde{b}$, for all $\lambda, 0 \leq \lambda \leq 1$.

3. A new role in fuzzy arithmetic and fuzzy ordering

A definition of the multiplication of two symmetric fuzzy numbers is given by Ganesan and Veeramani in [7] based on the *Extension Principle* (see [12]). However, fuzzy arithmetic and a fuzzy ordering role based on the given definition, is established by many researchers. Although many valuable works are appeared in the literature, there are a big limitation in the basic definition. In fact, the symmetric assumption is not reasonable in practice since there are many real situations that need to be dealt with by decision makers, especially when the main parameters of the system is formulated in the more general case and frankly in the form of non-symmetric fuzzy number. Moreover, by introducing a new definition of the length of ω , we may keep the length of the resulted fuzzy number which is obtained based on the given multiplication role.

For defining the multiplication of the two (non-symmetric) trapezoidal fuzzy numbers, we first needed to define the following notations:

Let

$$\theta = \{a_1b_1, a_1b_2, a_2b_1, a_2b_2\}, \quad \gamma = \text{Average } \theta.$$

Then, define

$$\alpha = |\gamma - \min\theta| \text{ and } \beta = |\max\theta - \gamma|.$$

Let $\omega = \left| \frac{\beta - \alpha}{2} \right|$ and we now may define the multiplication of two non-symmetric trapezoidal fuzzy numbers as an extended role of the given definition for the symmetric form of fuzzy numbers which is defined by Ganesan and Veeramani in [4], as follows:

Definition 3.1. Let $\tilde{a} = [a_1, a_2, h_1, h_2]$ and $\tilde{b} = [b_1, b_2, k_1, k_2]$ be two trapezoidal fuzzy numbers. Then the arithmetic operations on \tilde{a} and \tilde{b} are given by:

$$\begin{aligned} \text{Multiplication: } \tilde{a}\tilde{b} &= [a_1, a_2, h_1, h_2][b_1, b_2, k_1, k_2] \\ &= \left[\left(\frac{a_1 + a_2}{2} \right) \left(\frac{b_1 + b_2}{2} \right) - \omega, \left(\frac{a_1 + a_2}{2} \right) \left(\frac{b_1 + b_2}{2} \right) + \omega, |a_2k_1 + b_2h_1|, |a_2k_2 + b_2h_2| \right]. \end{aligned}$$

From the above definition, it is clear that

$$\lambda\tilde{a} = \begin{cases} [\lambda a_1, \lambda a_2, \lambda h_1, \lambda h_2], & \text{for } \lambda \geq 0, \\ [\lambda a_2, \lambda a_1, -\lambda h_2, -\lambda h_1], & \text{for } \lambda < 0. \end{cases}$$

Remark 2. Depending upon the need, we overcame the limitation of the multiplication role which is given just for the symmetric kind of trapezoidal fuzzy numbers. See in [4]

Definition 3.2. The model

$$\max \tilde{z} = \sum_{j=1}^n \tilde{c}_j \tilde{x}_j$$

(2-1)

$$S. t: \quad \sum_{j=1}^n a_{ij} \tilde{x}_j \leq \tilde{b}_i, \quad i = 1, \dots, m$$

$$\tilde{x}_j \geq \tilde{0}, \quad j = 1, \dots, n$$

if $a_{ij} \in \mathbb{R}$, and $\tilde{c}_j, \tilde{x}_j, \tilde{b}_i \in \mathcal{F}(\mathbb{R})$, $i = 1, \dots, m$, $j = 1, \dots, n$ is called a *Semi-Fuzzy Linear Programming (SFLP) problem*.

Definition 3.3. Any $\tilde{X} = (\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_n) \in \mathcal{F}^n(\mathbb{R})$, where each $\tilde{x}_i \in \mathcal{F}(\mathbb{R})$, which satisfies the constraints and non-negativity restrictions of (2-1) is said to be a *fuzzy feasible solution* to (2-1).

Definition 3.4. Let Q be the set of all fuzzy feasible solutions of (1). A fuzzy feasible solution $\tilde{X}_o \in Q$ is said to be a *fuzzy optimum solution* to (1), if $\tilde{C}\tilde{X}_o \geq \tilde{C}\tilde{X}$ for all $\tilde{X} \in Q$, where $\tilde{C} = (\tilde{c}_1, \tilde{c}_2, \dots, \tilde{c}_n)$ and $\tilde{C}\tilde{X} = \tilde{c}_1\tilde{x}_1 + \tilde{c}_2\tilde{x}_2 + \dots + \tilde{c}_n\tilde{x}_n$.

3.2. Fuzzy Basic feasible solution

The concept of fuzzy basic feasible solution is similar to the given definition in [8]. We give a new definition for fuzzy solution associated to the discussed model as below:

Definition 3.5. Let $\tilde{X} = (\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_n) \in \mathcal{F}^n(\mathbb{R})$, suppose that $\tilde{x} = (\tilde{x}_B^T, \tilde{x}_N^T)$, where $\tilde{x}_B \approx B^{-1}\tilde{b}, \tilde{x}_N \approx \tilde{0}$ to be a fuzzy basic feasible solution of the system $A\tilde{x} \approx \tilde{b}$. If $\tilde{x}_j =$

$[-\alpha_j, \alpha_j, \underline{h}_j, \overline{h}_j]$ for some $\alpha_i > 0$, \underline{h}_j and $\overline{h}_j \geq 0$ that is $\tilde{x}_j \approx \tilde{0}$ and every basic variable of the corresponding to every feasible basic B is positive, \tilde{x} is said to be a degenerated fuzzy basic feasible solution.

The following theorem concerns the so-called nondegenerate FNLP problems,

Theorem 3.1. Let the FNLP problem be nondegenerate. A basic feasible solution $\tilde{x}_B \approx B^{-1}\tilde{b}$, $\tilde{x}_N \approx \tilde{0}$ is optimal to (2-1) only if $\tilde{z}_j \geq \tilde{c}_j$ for all $j, 1 \leq j \leq n$.

Proof: Suppose that $\tilde{x}_* = (\tilde{x}_B^T, \tilde{x}_N^T)^T$ is a basic feasible solution to (2-1), where $\tilde{x}_B \approx B^{-1}\tilde{b}$, $\tilde{x}_N \approx \tilde{0}$. Then, $\tilde{z} = \tilde{c}_B \tilde{x}_B = \tilde{c}_B B^{-1}\tilde{b}$. On the other hand, for every feasible solution \tilde{x} , we have $\tilde{b} \approx A\tilde{x} \approx B\tilde{x}_B + N\tilde{x}_N$. Hence, we obtain:

$$\tilde{z} = \tilde{c}\tilde{x} - \sum_{j \neq B_i} (\tilde{z}_j - \tilde{c}_j)\tilde{x}_j. \quad (3-1)$$

Then,

$$\tilde{z} = \tilde{z}_* = \tilde{c}_B \tilde{x}_B + \tilde{c}_N \tilde{x}_N = \tilde{c}_B B^{-1}\tilde{b} - \sum_{j \neq B_i} (\tilde{c}_B B^{-1}a_j - \tilde{c}_j)\tilde{x}_j$$

The proof can now be completed using (3-1) and Theorem 3.2 given in Section 3. \square

Now we are going to devise a fuzzy primal simplex algorithm for solving the Problem (2-1).

3.3. Primal simplex method in tableau format for the fuzzy linear Problems

Consider the fuzzy linear programming problem as in (2-1). We rewrite the fuzzy linear programming problem as:

$$\begin{aligned} \text{Max } \tilde{z} &= \tilde{c}_B \tilde{x}_B + \tilde{c}_N \tilde{x}_N \\ \text{s. t. } & B\tilde{x}_B + N\tilde{x}_N = \tilde{b} \\ & \tilde{x}_B \geq \tilde{0}, \tilde{x}_N \geq \tilde{0}. \end{aligned}$$

Hence we have $\tilde{x}_B + B^{-1}N\tilde{x}_N \approx B^{-1}\tilde{b}$. Therefore, $\tilde{z} + (\tilde{c}_B B^{-1}N - \tilde{c}_N)\tilde{x}_N \approx \tilde{c}_B B^{-1}\tilde{b}$. With $\tilde{x}_N \approx \tilde{0}$ we have $\tilde{x}_B = B^{-1}\tilde{b} \approx \tilde{y}_0$, and $\tilde{z} \approx \tilde{c}_B B^{-1}\tilde{b}$. Thus we rewrote the above problem as in Table 1.

Table 1. The primal simplex tableau.

Basis	\tilde{x}_B	\tilde{x}_N	R.H.S
\tilde{z}	$\tilde{0}$	$\tilde{z}_N - \tilde{c}_N = \tilde{c}_B B^{-1}N - \tilde{c}_N$	$\tilde{y}_{00} = \tilde{c}_B B^{-1}\tilde{b}$
\tilde{x}_B	I	$Y = B^{-1}N$	$\tilde{y}_0 \approx B^{-1}\tilde{b}$

Remark 3. Table 1 gives all the information needed to proceed with the simplex method. The fuzzy cost row in Table 1 is $\tilde{y}_0^T = \tilde{c}_B B^{-1} A - \tilde{c}$, where $\tilde{y}_{0j} = \tilde{c}_B B^{-1} a_j - \tilde{c}_j = \tilde{z}_j - \tilde{c}_j, 1 \leq j \leq n$, with $\tilde{y}_{0j} \approx \tilde{0}$ for $j = B_i, 1 \leq i \leq m$. According to the optimality conditions (Theorem 3.1), we are at the optimal solution if $\tilde{y}_{0j} \geq \tilde{0}$ for all $j \neq B_i, 1 \leq i \leq m$. On the other hand, if $\tilde{y}_{0j} \leq \tilde{0}$ for some $k \neq B_i, 1 \leq i \leq m$, the problem is either unbounded or an exchange of a basic variable \tilde{x}_{B_r} and the nonbasic variable \tilde{x}_k can be made to increase the rank of the objective value (under nondegeneracy assumption). The following results established in [9] help us devise the fuzzy primal simplex algorithm.

Theorem 3.2. If there is a column k (not in basis) in a fuzzy primal simplex tableau as $\tilde{y}_{0k} = \tilde{z}_k - \tilde{c}_k < \tilde{0}$ and $y_{ik} \leq 0, 1 \leq i \leq m$, the problem (2-1) is unbounded.

Theorem 3.3. If a nonbasic index k exists in a fuzzy primal simplex tableau like $\tilde{y}_{0k} = \tilde{z}_k - \tilde{c}_k < \tilde{0}$ and there exists a basic index B_i like $y_{ik} \geq 0$, a pivoting row r can be found so that pivoting on y_{rk} can yield a feasible tableau with a corresponding nondecreasing (increasing under nondegeneracy assumption) fuzzy objective value.

Remark 4 (see [9]). If there exists k such that $\tilde{y}_{0k} < \tilde{0}$ and the problem is not unbounded, r can be chosen as

$$\frac{\tilde{y}_{ro}}{y_{rk}} = \min_{1 \leq i \leq m} \left\{ \frac{\tilde{y}_{io}}{y_{ik}}, y_{ik} > 0 \right\},$$

Where it is the minimum fuzzy value of the above ratio.

in order to replace \tilde{x}_{B_r} in the basis by \tilde{x}_k , resulting in a new basis $B = (a_{B_1}, a_{B_2}, \dots, a_{B_{r-1}}, a_k, a_{B_{r+1}}, \dots, a_{B_m})$. The new basis is primal feasible and the corresponding fuzzy objective value is nondecreasing (increasing under nondegeneracy assumption). It can be shown that the new simplex tableau is obtained by pivoting on y_{rk} , i.e. doing Gaussian elimination on the k th column by using the pivot row r , with the pivot y_{rk} , to transform the k th column to the unit vector e_r . It is easily seen that the new fuzzy objective value is: $\tilde{y}_{00} = \tilde{y}_{00} - \tilde{y}_{0k} \frac{y_{ro}}{y_{rk}} \geq \tilde{y}_{00}$ where $\tilde{y}_{0k} \leq \tilde{0}$ and $\frac{y_{ro}}{y_{rk}}$ (if the problem is nondegenerate, and consequently $\frac{y_{ro}}{y_{rk}} > 0$ and hence $\tilde{y}_{00} \approx \tilde{y}_{00}$).

We now describe the pivoting strategy.

3.4. Pivoting and change of basis

If \tilde{x}_k enters the basis and \tilde{x}_{B_r} leaves the basis, pivoting on y_{rk} in the primal simplex tableau is carried out as follows:

- 1) Divide row r by y_{rk} .
- 2) For $i = 0, 1, \dots, m$ and $i \neq r$, update the i th row by adding to it $-y_{ik}$ times the new r th row.

We now present the primal simplex algorithm for the FNLP problem.

3.5. The main steps of fuzzy primal simplex algorithm

Algorithm 3.1: The fuzzy primal simplex method

Assumption: A basic feasible solution with basis B and the corresponding simplex tableau is at hand.

1. The fuzzy basic feasible solution is given by $\tilde{x}_B = \tilde{y}_o = B^{-1}\tilde{b}$ and $\tilde{x}_N \approx \tilde{0}$. The fuzzy objective value is: $\tilde{z} = \tilde{y}_{oo} = \tilde{c}_B B^{-1}\tilde{b}$.

2. Calculate $\tilde{y}_{oj} = \tilde{z}_j - \tilde{c}_j, 1 \leq j \leq n$, with for $j \neq B_i, 1 \leq i \leq m$.

Let $\tilde{y}_{ok} = \min_{1 \leq j \leq n} \{\tilde{y}_{oj}\}$. If $\tilde{y}_{ok} \geq \tilde{0}$, then stop; the current solution is optimal.

3. If $\tilde{y}_{ok} \leq \tilde{0}$, then stop; the problem is unbounded. Otherwise, it determines an index r corresponding to a variable \tilde{x}_{B_r} leaving the basis as follows:

$$\frac{\tilde{y}_{ro}}{y_{rk}} = \min_{1 \leq i \leq m} \left\{ \frac{\tilde{y}_{io}}{y_{ik}}, y_{ik} > 0 \right\}.$$

4. Pivot on and update the simplex tableau. Go to step 2.

Remark 5 (limitation of existing method). Investigation of the current models and methods show that almost all of them are disable to formulate the real problems when the symmetric assumption is not at hand. Furthermore, by defining a new product role, we may to define a practical method for solving the generalized model, while the methods cannot work by the pioneering approach.

Now we are on the point of giving some illustrative examples. In the next subsection, we give two examples. The first problem is concerning to the symmetric form of fuzzy number that discussed by Ganesan et al. in [4]. This example shows how we can solve the existing shortcomings of the first model which is given in Example 3.1.

3.6. Numerical examples

Here, we are going to explore the proposed approach for solving a linear programming problem with symmetric fuzzy numbers which is considered by Ganesan et al. in [4].

Example 3.1. We consider the fuzzy mathematical model which is given by Ganesan et al. in [4]. The corresponding model is given in the below.

$$\max \tilde{Z} \approx [13, 15, 2, 2] \tilde{x}_1 + [12, 14, 3, 3] \tilde{x}_2 + [15, 17, 2, 2] \tilde{x}_3$$

$$\text{subject to } 12\tilde{x}_1 + 13\tilde{x}_2 + 12\tilde{x}_3 \leq [475, 505, 6, 6],$$

$$14\tilde{x}_1 + 13\tilde{x}_3 \leq [460, 480, 8, 8],$$

$$12\tilde{x}_1 + 15\tilde{x}_2 \leq [465, 495, 5, 5]$$

$$\tilde{x}_1 \geq \tilde{0}, \tilde{x}_2 \geq \tilde{0}, \tilde{x}_3 \geq \tilde{0}$$

Now the standardized form of the fuzzy linear programming problem becomes,

$$\max \tilde{Z} \approx [13, 15, 2, 2] \tilde{x}_1 + [12, 14, 3, 3] \tilde{x}_2 + [15, 17, 2, 2] \tilde{x}_3$$

$$\text{subject to } 12\tilde{x}_1 + 13\tilde{x}_2 + 12\tilde{x}_3 + \tilde{s}_1 \approx [475, 505, 6, 6],$$

$$14\tilde{x}_1 + 13\tilde{x}_3 + \tilde{s}_2 \approx [460, 480, 8, 8],$$

$$12\tilde{x}_1 + 15\tilde{x}_2 + \tilde{s}_3 \approx [465, 495, 5, 5]$$

$$\tilde{x}_1 \geq \tilde{0}, \tilde{x}_2 \geq \tilde{0}, \tilde{x}_3 \geq \tilde{0}$$

Using the multiplication role which is established by Ganesan et al., the fuzzy optimal solution and the fuzzy optimal value of objective function is as follows:

$$\tilde{x}_1 = [0, 0, 0, 0], \tilde{x}_2 = \left[\frac{415}{169}, \frac{1045}{169}, \frac{174}{169}, \frac{174}{169} \right], \tilde{x}_3 = \left[\frac{460}{13}, \frac{480}{13}, \frac{8}{13}, \frac{8}{13} \right],$$

$$\tilde{s}_1 = [0, 0, 0, 0], \tilde{s}_2 = [0, 0, 0, 0], \tilde{s}_3 = \left[\frac{62910}{169}, \frac{77430}{169}, \frac{3455}{169}, \frac{3455}{169} \right].$$

$$\tilde{Z}_V = \left[\frac{94235}{169}, \frac{120265}{169}, \frac{19819}{169}, \frac{19819}{169} \right].$$

Using a ranking function such as Yager, we obtain $R(\tilde{Z}_V) = \frac{107250}{169}$.

By substituting the fuzzy values of \tilde{x}_2 and \tilde{x}_3 in the objective function and using the new role of fuzzy multiplication, we have:

$$\tilde{Z}_H = \tilde{c}_1 \tilde{x}_1 + \tilde{c}_2 \tilde{x}_2 + \tilde{c}_3 \tilde{x}_3$$

$$\tilde{c}_1 \tilde{x}_1 = [13, 15, 2, 2][0, 0, 0, 0] = [0, 0, 0, 0]$$

$$\tilde{c}_2 \tilde{x}_2 = \left[\frac{415}{169}, \frac{1045}{169}, \frac{174}{169}, \frac{174}{169} \right][12, 14, 3, 3] = \left[\frac{9175}{169}, \frac{9805}{169}, \frac{5571}{169}, \frac{5571}{169} \right]$$

$$\begin{aligned} \tilde{c}_3 \tilde{x}_3 &= [15,17,2,2] \left[\frac{460}{13}, \frac{480}{13}, \frac{8}{13}, \frac{8}{13} \right] = \left[\frac{97630}{169}, \frac{97890}{169}, \frac{1096}{169}, \frac{1096}{169} \right] \\ \tilde{Z}_H &= [0,0,0,0] + \left[\frac{9175}{169}, \frac{9805}{169}, \frac{5571}{169}, \frac{5571}{169} \right] + \left[\frac{97630}{169}, \frac{97890}{169}, \frac{14248}{169}, \frac{14248}{169} \right] \\ &= \left[\frac{106805}{169}, \frac{107695}{169}, \frac{19819}{169}, \frac{19819}{169} \right] \end{aligned}$$

We clearly obtain $R(\tilde{Z}_H) = \frac{107250}{169}$.

So, according to the ordering role which is given in Definition 2.2, it is concluded that

$$\tilde{Z}_V \approx \tilde{Z}_H$$

In fact, it has been shown ,in this example, that the proposed method can solve the symmetric version of the given fuzzy numbers as well as Ganesan’s method.

Example 3.2 In this example, we consider a general form of trapezoidal fuzzy number for coefficients in the objective function, if it is not necessary to be symmetric.

$$\max \tilde{Z} \approx [13, 15, 3, 4] \tilde{x}_1 + [12, 14, 4, 5] \tilde{x}_2 + [15, 17, 3, 4] \tilde{x}_3$$

$$\text{subject to } 12\tilde{x}_1 + 13\tilde{x}_2 + 12\tilde{x}_3 \leq [475, 505, 6, 6],$$

$$14\tilde{x}_1 + 13\tilde{x}_3 \leq [460, 480, 8, 8],$$

$$12\tilde{x}_1 + 15 \tilde{x}_2 \leq [465, 495, 5, 5]$$

$$\tilde{x}_1 \geq \tilde{0}, \tilde{x}_2 \geq \tilde{0}, \tilde{x}_3 \geq \tilde{0}$$

Now the standard form of the fuzzy linear programming problem becomes

$$\max \tilde{Z} \approx [13, 15, 3, 4] \tilde{x}_1 + [12, 14, 4, 5] \tilde{x}_2 + [15, 17, 3, 4] \tilde{x}_3$$

$$\text{subject to } 12\tilde{x}_1 + 13\tilde{x}_2 + 12\tilde{x}_3 + \tilde{x}_4 \approx [475, 505, 6, 6],$$

$$14\tilde{x}_1 + 13\tilde{x}_3 + \tilde{x}_5 \approx [460, 480, 8, 8],$$

$$12\tilde{x}_1 + 15 \tilde{x}_2 + \tilde{x}_6 \approx [465, 495, 5, 5],$$

$$\tilde{x}_1 \geq \tilde{0}, \tilde{x}_2 \geq \tilde{0}, \tilde{x}_3 \geq \tilde{0}.$$

The first tableau of the fuzzy primal simplex algorithm is given as below:

Table 2. The first simplex tableau.

Basic	\tilde{x}_1	\tilde{x}_2	\tilde{x}_3	\tilde{x}_4	\tilde{x}_5	\tilde{x}_6	R.H.S.
Z	(-15,-13,4,3)	(-14,-12,5,4)	(-17,-15,4,3)	$\tilde{0}$	$\tilde{0}$	$\tilde{0}$	$\tilde{0}$
\tilde{x}_4	12	13	12	1	0	0	(475,505,6,6)
\tilde{x}_5	14	0	13	0	1	0	(460,480,8,8)
\tilde{x}_6	12	15	0	0	0	1	(465,495,5,5)

The first Since $(\tilde{y}_{01}, \tilde{y}_{02}, \tilde{y}_{03}) = ((-15, -13, 4, 3), (-14, -12, 5, 4), (-17, -15, 4, 3))$, and $(\mathcal{R}(\tilde{y}_{01}), \mathcal{R}(\tilde{y}_{02}), \mathcal{R}(\tilde{y}_{03})) = (-14.25, -13.25, -16.25)$, then \tilde{x}_3 enters the basis and based on the minimum ratio test, the leaving variable is \tilde{x}_5 . Pivoting on $y_{53} = 13$.

After calculating the amount of R.H.S. column in Table 1, it has been found that the multiplication role which was defined by Ganesan and Veeramani in [4] is not satisfying and we must hence use Definition 2.2 for obtaining the amount of multiplication as follows:

$$y_{00}^{new} = \tilde{0} + (\frac{460}{13}, \frac{480}{13}, \frac{8}{13}, \frac{8}{13})(15, 17, 3, 4)$$

Now, assume that $\tilde{a} = (\frac{460}{13}, \frac{480}{13}, \frac{8}{13}, \frac{8}{13})$ and $\tilde{b} = (15, 17, 3, 4)$, then

$$\theta = \{a_1b_1, a_1b_2, a_2b_1, a_2b_2\} = \left\{ \frac{6900}{13}, \frac{7820}{13}, \frac{7200}{13}, \frac{8160}{13} \right\},$$

$$\gamma = \text{Average } \theta = \frac{7520}{13}.$$

$$\alpha = |\gamma - \min\theta| = \frac{620}{13}, \quad \beta = |\max\theta - \gamma| = \frac{640}{13}.$$

So, $\omega = \left| \frac{\beta - \alpha}{2} \right| = \frac{10}{13}.$

$$\begin{aligned} \tilde{a}\tilde{b} &= \left[\left(\frac{a_1 + a_2}{2} \right) \left(\frac{b_1 + b_2}{2} \right) - \omega, \left(\frac{a_1 + a_2}{2} \right) \left(\frac{b_1 + b_2}{2} \right) + \omega, |a_2k_1 + b_2h_1|, |a_2k_2 + b_2h_2| \right] \\ &= \left[\frac{7510}{13}, \frac{7530}{13}, \frac{1576}{13}, \frac{2056}{13} \right]. \end{aligned}$$

Updating the data is based on the about calculation concluded in Table 2:

Table 3. The second simplex tableau.

Basic	\tilde{x}_1	\tilde{x}_2	\tilde{x}_3	\tilde{x}_4	\tilde{x}_5	\tilde{x}_6	R.H.S.
Z	$(\frac{15}{13}, \frac{69}{13}, \frac{94}{13}, \frac{95}{13})$	$(-14, -12, 5, 4)$	$\tilde{0}$	$\tilde{0}$	$(\frac{15}{13}, \frac{17}{13}, \frac{3}{13}, \frac{4}{13})$	$\tilde{0}$	$(\frac{7510}{13}, \frac{7530}{13}, \frac{1576}{13}, \frac{2056}{13})$
\tilde{x}_4	$\frac{-12}{13}$	13	0	1	$\frac{-12}{13}$	0	$(\frac{415}{13}, \frac{1045}{13}, \frac{174}{13}, \frac{174}{13})$
\tilde{x}_3	$\frac{14}{13}$	0	1	0	$\frac{1}{13}$	0	$(\frac{460}{13}, \frac{480}{13}, \frac{8}{13}, \frac{8}{13})$
\tilde{x}_6	$\frac{13}{12}$	15	0	0	0	1	$(465, 495, 5, 5)$

Since the problem is maximization, the optimality condition is not valid and hence \tilde{x}_2 enters the basis and the leaving variable is \tilde{x}_4 . The next tableau based on the following calculations is shown in Table 3.

$$y_{oo}^{new} = \left(\frac{7510}{13}, \frac{7530}{13}, \frac{1576}{13}, \frac{2056}{13}\right) + \left(\frac{415}{169}, \frac{1045}{169}, \frac{174}{169}, \frac{174}{169}\right)(12,14,4,5).$$

We suppose that $\tilde{a} = \left(\frac{415}{169}, \frac{1045}{169}, \frac{174}{169}, \frac{174}{169}\right)$ and $\tilde{b} = (12,14,4,5)$, then

$$\theta = \{a_1b_1, a_1b_2, a_2b_1, a_2b_2\} = \left\{\frac{4980}{169}, \frac{12540}{169}, \frac{5810}{169}, \frac{14630}{169}\right\},$$

$$\gamma = \text{Average } \theta = \frac{9490}{169},$$

$$\alpha = |\gamma - \min\theta| = \frac{4510}{169}, \quad \beta = |\max\theta - \gamma| = \frac{5140}{169}.$$

Hence,

$$\omega = \left| \frac{\beta - \alpha}{2} \right| = \frac{315}{169}.$$

Then,

$$\tilde{a}\tilde{b} = \left[\binom{a_1+a_2}{2} \binom{b_1+b_2}{2} - \omega, \binom{a_1+a_2}{2} \binom{b_1+b_2}{2} + \omega, |a_2k_1 + b_2h_1|, |a_2k_2 + b_2h_2| \right] = \left[\frac{18600}{169}, \frac{19230}{169}, \frac{6616}{169}, \frac{7661}{169} \right].$$

Then, the objective value in this basis is as follows:

$$\tilde{a}\tilde{b} + \left(\frac{7510}{13}, \frac{7530}{13}, \frac{1576}{13}, \frac{2056}{13}\right) = \left[\frac{116230}{169}, \frac{117120}{169}, \frac{27104}{169}, \frac{34389}{169} \right].$$

Table 4. The optimal simplex tableau.

Basic	\tilde{x}_1	\tilde{x}_2	\tilde{x}_3	\tilde{x}_4	\tilde{x}_5	\tilde{x}_6	R.H.S.
Z	$\left(\frac{27}{169}, \frac{753}{169}, \frac{1282}{169}, \frac{1283}{169}\right)$	0	0	$\left(\frac{12}{13}, \frac{14}{13}, \frac{4}{13}, \frac{5}{13}\right)$	$\left(\frac{27}{169}, \frac{77}{169}, \frac{99}{169}, \frac{100}{169}\right)$	0	$\left(\frac{116230}{169}, \frac{117120}{169}, \frac{27104}{169}, \frac{34389}{169}\right)$
\tilde{x}_2	$\frac{-12}{169}$	1	0	$\frac{1}{13}$	$\frac{-12}{169}$	0	$\left(\frac{415}{169}, \frac{1045}{169}, \frac{174}{169}, \frac{174}{169}\right)$
\tilde{x}_3	$\frac{14}{13}$	0	1	0	$\frac{1}{13}$	0	$\left(\frac{460}{13}, \frac{480}{13}, \frac{8}{13}, \frac{8}{13}\right)$
\tilde{x}_6	$\frac{2208}{169}$	0	0	-15	$\frac{180}{169}$	1	$\left(\frac{62910}{169}, \frac{77430}{169}, \frac{1019}{169}, \frac{1019}{169}\right)$

This table is optimal and the optimal value of the variables and the objective function are as follows:

$$\tilde{x}_2^* = \left(\frac{415}{169}, \frac{1045}{169}, \frac{174}{169}, \frac{174}{169}\right), \tilde{x}_3^* = \left(\frac{460}{13}, \frac{480}{13}, \frac{8}{13}, \frac{8}{13}\right), \tilde{x}_6^* = \left(\frac{62910}{169}, \frac{77430}{169}, \frac{1019}{169}, \frac{1019}{169}\right)$$

$$\text{and } \tilde{x}_1^* = \tilde{x}_4^* = \tilde{x}_5^* = \tilde{0}, \text{ and } Z^* = \left(\frac{116230}{169}, \frac{117120}{169}, \frac{27104}{169}, \frac{34389}{169}\right).$$

Where

$$\begin{aligned} Z_H &= [13, 15, 3, 4]\tilde{x}_1^* + [12, 14, 4, 5]\tilde{x}_2^* + [15, 17, 3, 4]\tilde{x}_3^* \\ &= [13, 15, 3, 4]\tilde{0} + (12, 14, 4, 5)\left(\frac{415}{169}, \frac{1045}{169}, \frac{174}{169}, \frac{174}{169}\right) + (15, 17, 3, 4)\left(\frac{460}{13}, \frac{480}{13}, \frac{8}{13}, \frac{8}{13}\right) \\ &= \left(\frac{116230}{169}, \frac{117120}{169}, \frac{27104}{169}, \frac{34389}{169}\right) \end{aligned}$$

Thus, it has been seen that the proposed approach gave the same results as the mentioned problem in the method proposed by Ganesan et al. In particular, the proposed arithmetic allows the decision makers to model as a general type, where the promoters can be formulated as a general form of trapezoidal fuzzy numbers which is more appropriate for real situations than just in the symmetric form.

4. Conclusion

In the paper, a new role for the multiplication of two general forms of trapezoidal fuzzy numbers has been defined. In particular, we saw that the new model sounds to be more appropriate for the real situation, while in the pioneering model which was suggested by Ganessan et al., [4] and subsequently Das et al. [1], the trapezoidal fuzzy numbers are assumed to be essential in the symmetric form. Therefore, these tools can be useful for preparing some solving algorithms, fuzzy primal, dual simplex algorithms and so on.

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